THE GENERATION OF GRAVITATIONAL WAVES.
I. WEAK-FIELD SOURCES: A "PLUG-IN-AND-GRIND" FORMALISM*

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ABSTRACT

This paper derives (§§II-IV) and summarizes (§VI) a new "plug-in-and-grind" formalism for calculating the gravitational waves emitted by any system with weak internal gravitational fields. If the internal fields have negligible influence on the system's motions, then the new formalism reduces to standard "linearized theory". Whether or not gravity affects the motions, if the motions are slow and internal stresses are weak, then the new formalism reduces to the standard "quadrupole-moment formalism" (§V). In the general case the new formalism expresses the radiation in terms of a retarded Green's function for slightly curved spacetime, and then breaks the Green's-function integral into five easily understood pieces: <u>direct radiation</u>, produced directly by the motions of the source; <u>whump radiation</u>, produced by the "gravitational stresses" of the source; <u>transition radiation</u>, produced by a time-changing time delay ("Shapiro effect") in the propagation of the nonradiative, 1/r field of the source; <u>focussing radiation</u>, produced when one portion of the source focusses, in a time-dependent way, the nonradiative field of another portion of the source, and <u>tail radiation</u>, produced by "backscatter" of the nonradiative field in regions of focussing.

I. INTRODUCTION

a) Introduction to This Series of Papers

Thanks to the pioneering work of Joseph Weber (1960,1969), "gravitational-wave astronomy" may be a reality by 1980. Although Weber's "events" may turn out to be non-gravitational in origin, second-generation detectors of the Weber "resonant-bar" type, with amplitude sensitivities roughly 100-fold better than today's bars, are now under construction (Braginsky 1974, Fairbank and Hamilton, as described in Boughn et al. 1974); and third-generation detectors are being discussed. The third generation should be able to detect and study the gravitational-wave bursts generated several times per year by supernovae in the Virgo cluster of galaxies. Detectors with other designs may succeed in detecting waves from pulsars [see, e.g., Braginsky and Nazarenko (1971)] and from near-encounters of stars in dense star clusters [gravitational bremsstrahlung; see. e.g., Zel'dovich and Polnarev (1974)]. And, of course, totally unexpected sources may be detected. [For reviews of the prospects for gravitational-wave astronomy see Misner (1974), Rees (1974), and Press and Thorne (1972).]

In preparation for the era of gravitational-wave astronomy, our Caltech research group has embarked on a new project: We seek (1) to elucidate the realms of validity of the standard wave-generation formulas; (2) to devise new techniques for calculating gravitational-wave generation with new realms of validity; and (3) to calculate the waves generated by particular models of astrophysical systems. Throughout this project we shall confine ourselves to general relativity theory.

Most past calculations of gravitational-wave generation use one of three formalisms: (1) "linearized theory" or its quantum-theory analogue;

(2) the "quadrupole-moment formalism"; (3) "first-order perturbations of stationary, fully relativistic spacetimes."

"Linearized theory" is the formalism obtained by linearizing general relativity about flat spacetime [see, e.g., Chapters 18 and 35 of Misner, Thorne, and Wheeler (1973)--cited henceforth as "MTW"]. It is also the unique linear spin-two field theory of gravitation in flat spacetime--and as such it has a simple quantum-theory formulation. [For references and overview see, in MTW, §7.1, Box 7.1, and part 5 of Box 17.2]. Linearized theory is typically used to calculate wave generation when the source's self gravity has negligible influence on its motions (e.g., waves from spinning rods and from electromagnetic fields in a cavity). In this paper we shall devise a new wave-generation formalism valid for any system with small but non-negligible self gravity; and in Paper III (Kovács and Thorne 1975) we shall use that formalism to calculate the gravitational bremsstrahlung produced when two stars fly past each other with large impact parameter, but with arbitrary relative masses and velocities.

The "quadrupole-moment formalism" (in which the wave amplitude is proportional to the second time derivative of the source's mass quadrupole moment) dates back to Einstein (1918), and has been canonized by Landau and Lifshitz (1951). The derivations of this formalism which we find in the literature are valid only for systems with slow internal motions and weak (but non-negligible) internal gravitational fields [see, e.g., the post$^{5/2}$-Newtonian derivation by Chandrasekhar and Esposito (1970), the matched-asymptotic-expansion derivation by Burke (1971), and the de Donder-gauge derivation by Landau and Lifshitz (1951) as made more explicit in Chapter 36 of MTW]. However, a detailed analysis given in Paper II (Thorne 1975) shows that only the slow-motion assumption is needed: the quadrupole-moment formalism is valid for any slow-motion system, regardless of its internal field strengths. Paper II also extends that formalism to include the radiation produced by all of the source's other moments (both "mass" moments and "current" moments); and it derives formulas in terms of the moments for the near-zone fields, the radiation field, the radiation reaction, and the energy, momentum, and angular momentum carried off by the waves. In a forthcoming paper Thorne and Żytkow (1975) will use the extended formalism of Paper II to calculate the "current-quadrupole" gravitational waves produced by torsional oscillations of neutron stars.

"First-order perturbations of stationary, fully relativistic spacetimes" is a technique that has been used extensively in recent years to analyze waves from "fast-motion" oscillations of black holes and neutron stars, and from particles moving in the Schwarzschild and Kerr gravitational fields. [For reviews, see Press (1974), Ruffini (1973) and §36.5 of MTW; see also the recent paper by Chung (1973).] It is not yet clear whether our project will delve into this technique.

b) Overview of This Paper

In this paper we confine attention to systems with weak internal gravitational fields. Section II rewrites the exact Einstein field equations in a non-covariant form ("de Donder form") that is amenable to weak-field approximations. Section III gives a systematic account of approximate, weak-field formalisms based on the exact de Donder form of the field equations--including the accuracy of the various formalisms and their relationships to each other. Section IVa applies the analysis of Section III to astrophysical systems, and concludes that, when analyzing their structure and evolution, one must typically calculate the stress-energy tensor ${}_2T^{\mu\nu}$ and gravitational field ${}_1\bar{h}^{\mu\nu}$ with accuracies:

$$|(\text{error in } {}_2T^{\mu\nu}) / {}_2T^{00}| \lesssim \varepsilon^2$$

$$|(\text{error in } {}_1\bar{h}^{\mu\nu}) / {}_1\bar{h}^{00}| \lesssim \varepsilon$$

$$\varepsilon \equiv (\text{typical value of } {}_1\bar{h}^{00} \text{ inside source})$$

$$\sim (\text{mass of source})/(\text{size of source}).$$

Section IVa also concludes that the external gravitational field ${}_2\bar{h}^{\mu\nu}$ must typically be calculated to accuracy

$$|(\text{error in } {}_2\bar{h}^{\mu\nu}) / {}_2\bar{h}^{00}| \lesssim \varepsilon^2$$

if one desires reasonable accuracy in the radiative part of that field.

Section IVb presents a "post-linear" formalism for calculating a system's structure and evolution (${}_2T^{\mu\nu}$ and ${}_1\bar{h}^{\mu\nu}$) to the desired accuracy; and Section IVc derives a plug-in-and-grind formula for the higher-accuracy external field (${}_2\bar{h}^{\mu\nu}$), which contains the radiation. Section V shows how the resulting formalism, when applied to slow-motion systems, reduces to the standard "quadrupole-moment formalism".

We recommend that, before tackling the rest of this paper, the reader peruse Section VI. That section summarizes our post-linear formalism and our formula for the external (radiation) field.

The "guts" of this paper, in terms of complex calculations, reside in the Green's-function manipulations of Section IVc. Our particular way of handling the Green's functions is motivated in Appendix A, and has been influenced by the following papers: De Witt and Brehme (1960) (exact Green's functions for scalar and vector wave equations in curved spacetime); Robaschik (1963) (exact Green's function for tensor wave equation in curved spacetime); John (1973a,b), Bird (1974) and especially Peters (1966) (Green's functions in weakly curved spacetime). Although these papers had much influence on us, our specific manipulations are so different that we have found it impossible to trace the details of that influence in our writeup.

II. EXACT GENERAL RELATIVITY, REWRITTEN IN "WEAK-FIELD LANGUAGE"

We begin by writing the exact, nonlinear Einstein field equations in an arbitrary coordinate system in the form [§20.3 of MTW; §100 of Landau and Lifshitz (1962)]

$$H^{\mu\alpha\nu\beta}_{L-L\ ,\alpha\beta} = 16\pi(-g)(T^{\mu\nu} + t^{\mu\nu}_{L-L}) \tag{1}$$

where "L-L" means "Landau-Lifshitz", and where

$$H^{\mu\alpha\nu\beta}_{L-L} \equiv \mathfrak{g}^{\mu\nu}\mathfrak{g}^{\alpha\beta} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\mu\beta} \quad ; \tag{2a}$$

$$\mathfrak{g}^{\mu\nu} \equiv (-g)^{1/2} g^{\mu\nu}, \qquad (-g) = -\det\|g_{\mu\nu}\| = -\det\|\mathfrak{g}^{\mu\nu}\| \quad ; \tag{2b}$$

$$\begin{aligned} t^{\alpha\beta}_{L-L} \equiv [16\pi(-g)]^{-1}\{&\mathfrak{g}^{\alpha\beta}{}_{,\lambda}\mathfrak{g}^{\lambda\mu}{}_{,\mu} - \mathfrak{g}^{\alpha\lambda}{}_{,\lambda}\mathfrak{g}^{\beta\mu}{}_{,\mu} + \tfrac{1}{2} g^{\alpha\beta} g_{\lambda\mu}\mathfrak{g}^{\lambda\nu}{}_{,\rho}\mathfrak{g}^{\rho\mu}{}_{,\nu} \\ &- (g^{\alpha\lambda}g_{\mu\nu}\mathfrak{g}^{\beta\nu}{}_{,\rho}\mathfrak{g}^{\mu\rho}{}_{,\lambda} + g^{\beta\lambda}g_{\mu\nu}\mathfrak{g}^{\alpha\nu}{}_{,\rho}\mathfrak{g}^{\mu\rho}{}_{,\lambda}) + g_{\lambda\mu}g^{\nu\rho}\mathfrak{g}^{\alpha\lambda}{}_{,\nu}\mathfrak{g}^{\beta\mu}{}_{,\rho} \qquad (2c) \\ &+ \tfrac{1}{8}(2g^{\alpha\lambda}g^{\beta\mu} - g^{\alpha\beta}g^{\lambda\mu})(2g_{\nu\rho}g_{\sigma\tau} - g_{\rho\sigma}g_{\nu\tau})\mathfrak{g}^{\nu\tau}{}_{,\lambda}\mathfrak{g}^{\rho\sigma}{}_{,\mu}\} \ . \qquad (2) \end{aligned}$$

The equations of motion for the material stress-energy tensor $T^{\mu\nu}$ follow directly from the field equations (1) and can be written in the equivalent forms

$$T^{\alpha\beta}{}_{;\beta} = 0 \ , \qquad [(-g)(T^{\alpha\beta} + t^{\alpha\beta}_{L-L})]_{,\beta} = 0 \quad . \tag{3}$$

[Here and throughout this series of papers we use the notation and sign conventions of MTW; in particular $c = G = 1$; $g_{\alpha\beta}$ and $g^{\alpha\beta}$ are the components of the metric; commas and ∂'s denote partial derivatives, $Y_{,\alpha} = \partial_\alpha Y = \partial Y/\partial x^\alpha$; semicolons denote covariant derivatives with respect to the metric $g_{\alpha\beta}$; and our signature is $(-+++)$.]

Now, and henceforth in this paper, we impose three restrictions on our analysis: (1) We confine attention to systems with "weak internal gravity"--i.e., to systems throughout which one can introduce nearly Lorentz coordinates. (2) We confine ourselves to "isolated systems"--i.e., to systems that are surrounded by a region ("local wave zone"), much larger than a characteristic wavelength of the emitted waves, in which all waves are outgoing and in which external masses have negligible influence on the gravitational field. (3) We restrict our analysis to the interior of the source and its local wave zone, and throughout these regions we use nearly Lorentz, asymptotically flat coordinates, specialized to satisfy the deDonder gauge condition.

Mathematically, these restrictions state that the "gravitational field"

$$\bar{h}^{\mu\nu} \equiv -\mathfrak{g}^{\mu\nu} + \eta^{\mu\nu} \tag{4}$$

has the properties

$$|\bar{h}^{\mu\nu}| << 1 \quad \text{everywhere,} \tag{5a}$$

$$|\bar{h}^{\mu\nu}| \sim 1/r \quad \text{as} \quad r \to \infty, \text{ where } r = (x^2 + y^2 + z^2)^{1/2}, \tag{5b}$$

$$\bar{h}^{\mu\nu} \quad \text{is devoid of incoming waves at} \quad r \to \infty, \tag{5c}$$

$$\bar{h}^{\mu\nu}{}_{,\nu} = 0 \quad \text{(deDonder condition).} \tag{5d}$$

With these restrictions, the exact Einstein field equations (1) take on the form

$$\Box_s \bar{h}^{\mu\nu} = -16\pi(-g)^{1/2}(T^{\mu\nu} + t^{\mu\nu}_{L-L}) - (-g)^{-1/2}\bar{h}^{\mu\alpha}{}_{,\beta}\, \bar{h}^{\nu\beta}{}_{,\alpha} \; . \tag{6}$$

Here $\Box_s$ is the wave operator for scalar fields in the curved spacetime described by the metric $g_{\alpha\beta}$:

$$\Box_s \equiv (-g)^{-1/2}\, \partial_\alpha \, [(-g)^{1/2}\, g^{\alpha\beta}\partial_\beta] \quad . \tag{7}$$

[Appendix A, which is best read after one has finished reading the rest of the paper, explains why we write the field equations in terms of $\Box_s$ rather than in terms of some other wave operator such as $\Box_f \equiv \eta^{\alpha\beta}\partial_\alpha\partial_\beta$ (the flat-space wave operator) or $\Box_t$ (the curved-space wave operator for tensor fields).]

Equations (2b)-(7) are the exact, nonlinear equations of general relativity for any isolated, weak-field system--but they are written in a very special coordinate system rather than in generally covariant form.

Because $|\bar{h}^{\mu\nu}| << 1$, we can express each quantity in our formalism, except $T^{\mu\nu}$, as a power series in $\bar{h}^{\mu\nu}$. When writing down such a power series, it is convenient to raise and lower indices of $\bar{h}^{\mu\nu}$ with the Minkowski metric $\eta_{\alpha\beta} = \eta^{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$:

$$\bar{h}_\alpha{}^\nu \equiv \eta_{\alpha\mu}\bar{h}^{\mu\nu} \; , \qquad \bar{h}_{\alpha\beta} \equiv \eta_{\alpha\mu}\eta_{\beta\nu}\bar{h}^{\mu\nu} \; , \qquad \bar{h} \equiv \bar{h}_\alpha{}^\alpha \; , \qquad \text{etc.} \tag{8a}$$

It is also convenient to define a "trace-reversed" gravitational field $h^{\mu\nu}$ by

$$h^{\mu\nu} \equiv \bar{h}^{\mu\nu} - \frac{1}{2}\,\bar{h}\,\eta^{\mu\nu} \; , \tag{8b}$$

and to raise and lower its indices, like those of $\bar{h}^{\mu\nu}$, with the Minkowski metric. Note that equation (8b) implies

$$h \equiv h_{\alpha}{}^{\alpha} = -\bar{h} \;, \quad \bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\, h\, \eta^{\mu\nu} \quad . \tag{8c}$$

To derive the explicit power series expansions for $\mathfrak{g}^{\mu\nu}$, $g^{\mu\nu}$, $g_{\mu\nu}$, etc., one can proceed as follows. Equation (4) is the desired expansion for $\mathfrak{g}^{\mu\nu}$. It contains only two terms:

$$\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} - \bar{h}^{\mu\nu} \quad . \tag{9a}$$

The expansion for the metric determinant (-g) is obtained by inserting expression (9a) into the second of equations (2b):

$$\begin{aligned}(-g) &= -\det\|\mathfrak{g}^{\mu\nu}\| = -\det\|\eta^{\mu\nu} - \bar{h}^{\mu\nu}\| \\ &= 1 - \bar{h} + \frac{1}{2}\,[(\bar{h})^2 - \bar{h}^{\alpha\beta}\bar{h}_{\alpha\beta}] + 0[(\bar{h})^3] \quad .\end{aligned} \tag{9b}$$

The contravariant components of the metric are then obtained by inserting (9a,b) into the first of equations (2b)

$$\begin{aligned}g^{\mu\nu} &= (-g)^{-1/2}\, \mathfrak{g}^{\mu\nu} \\ &= \eta^{\mu\nu} - (\bar{h}^{\mu\nu} - \frac{1}{2}\,\bar{h}\,\eta^{\mu\nu} - \frac{1}{2}\,\bar{h}\,\bar{h}^{\mu\nu} + \frac{1}{8}\,\eta^{\mu\nu}[\,(\bar{h})^2 + 2\bar{h}^{\alpha\beta}\bar{h}_{\alpha\beta}] + 0[(\bar{h})^3]\end{aligned} \tag{9c}$$

and the covariant components are obtained as the matrix inverse of these contravariant components:

$$g_{\mu\nu} = \eta_{\mu\nu} + \bar{h}_{\mu\nu} - \frac{1}{2}\,\bar{h}\,\eta_{\mu\nu} + \bar{h}_{\mu\alpha}\bar{h}^{\alpha}{}_{\nu} - \frac{1}{2}\,\bar{h}\,\bar{h}_{\mu\nu} + \frac{1}{8}\,\eta_{\mu\nu}(\bar{h}^2 - 2\bar{h}^{\alpha\beta}\bar{h}_{\alpha\beta}) + 0[(\bar{h})^3]\,. \tag{9d}$$

The connection coefficients $\Gamma^{\mu}{}_{\alpha\beta}$, which appear in the usual expression $T^{\mu\nu}{}_{;\nu} = 0$ for the equations of motion, are obtained by inserting expansions

(9c,d) into the standard formula

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2} g^{\mu\gamma}(g_{\gamma\alpha,\beta} + g_{\gamma\beta,\alpha} - g_{\alpha\beta,\gamma})$$

$$= \frac{1}{2}(\bar{h}^{\mu}{}_{\alpha,\beta} + \bar{h}^{\mu}{}_{\beta,\alpha} - \bar{h}_{\alpha\beta}{}^{,\mu}) - \frac{1}{4}(\delta^{\mu}{}_{\alpha}\,\bar{h}_{,\beta} + \delta^{\mu}{}_{\beta}\,\bar{h}_{,\alpha} - \eta_{\alpha\beta}\bar{h}^{,\mu}) + O[(\bar{h})^2] \qquad (9e)$$

Similarly, the scalar-wave operator $\square_s$ is obtained by inserting expansions (9b,c) into equation (7), and using the deDonder gauge condition (5d) to simplify:

$$\square_s = \eta^{\alpha\beta}\,\partial_\alpha\partial_\beta - (\bar{h}^{\alpha\beta} - \frac{1}{2}\,\bar{h}\,\eta^{\alpha\beta})\partial_\alpha\partial_\beta + O[(\bar{h})^2] \quad ; \qquad (9f)$$

and the components of the Landau-Lifshitz pseudotensor are obtained by inserting expansions (9a,b,c,d) and the deDonder condition (5d) into equation (2c):

$$\begin{aligned} t^{\alpha\beta}_{L-L} = (16\pi)^{-1} \{ & \frac{1}{2}\,\eta^{\alpha\beta}\eta_{\lambda\mu}\bar{h}^{\lambda\nu}{}_{,\rho}\,\bar{h}^{\rho\mu}{}_{,\nu} + \eta_{\lambda\mu}\eta^{\nu\rho}\,\bar{h}^{\alpha\lambda}{}_{,\nu}\,\bar{h}^{\beta\mu}{}_{,\rho} \\ & - (\eta^{\alpha\lambda}\eta_{\mu\nu}\,\bar{h}^{\beta\nu}{}_{,\rho}\,\bar{h}^{\mu\rho}{}_{,\lambda} + \eta^{\beta\lambda}\eta_{\mu\nu}\,\bar{h}^{\alpha\nu}{}_{,\rho}\,\bar{h}^{\mu\rho}{}_{,\lambda}) \\ & + \frac{1}{8}\,(2\eta^{\alpha\lambda}\eta^{\beta\mu} - \eta^{\alpha\beta}\eta^{\lambda\mu})(2\eta_{\nu\rho}\eta_{\sigma\tau} - \eta_{\rho\sigma}\eta_{\nu\tau})\,\bar{h}^{\nu\tau}{}_{,\lambda}\,\bar{h}^{\rho\sigma}{}_{,\mu}\} + O[(\bar{h})^3] \;. \end{aligned} \qquad (9g)$$

Henceforth in this paper we shall regard $\mathfrak{g}^{\mu\nu}$, $(-g)$, $g^{\mu\nu}$, $g_{\mu\nu}$, $\Gamma^{\mu}{}_{\alpha\beta}$, $\square_s$, and $t^{\alpha\beta}_{L-L}$ as shorthand notation for the <u>infinite</u> power series expansions, whose first few terms are shown in equations (9). Given these expansions, the full content of general relativity is embodied in the equations of motion for the matter

If the system is several lumps with masses m and sizes ℓ, separated by distances $b \gg \ell$ (e.g., a binary star system or two stars flying past each other), then

$\varepsilon \sim m/b$ if one is interested only in the relative motions of the lumps

$\varepsilon \sim m/\ell$ if one is also interested in the internal structure and dynamics of the lumps.

We shall characterize every weak-field approximation formalism by two integers n_T and n_h. These "order indices" tell us the magnitude of the errors made by the formalism:[2]

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[2]Note that all of the $|T^{\mu\nu}|$ are $\lesssim T^{00}$, and consequently all of the $|\bar{h}^{\mu\nu}|$ are $\lesssim \bar{h}^{00}$. This fact dictates the form of equations (12).

$$|(\text{errors in } T^{\mu\nu}) / T^{00}| \sim \varepsilon^{n_T} \tag{12a}$$

$$|(\text{errors in } \bar{h}^{\mu\nu}) / \bar{h}^{00}| \sim \varepsilon^{n_h} \quad . \tag{12b}$$

For example, a formalism of order $(n_T, n_h) = (1,1)$ makes fractional errors of order ε in both the stress-energy tensor and the gravitational field, while a formalism of order (2,1) makes fractional errors ε^2 in $T^{\mu\nu}$ and ε in $\bar{h}^{\mu\nu}$.

Errors in $\bar{h}^{\mu\nu}$, when fed into the equations of motion (10a), produce errors in $T^{\mu\nu}$; and similarly, errors in $T^{\mu\nu}$, when fed into the field equations (10b), produce errors in $\bar{h}^{\mu\nu}$. This feeding process places constraints on the order indices (n_T, n_h) of any self-consistent approximation formalism. The constraints are revealed explicitly by an order-of-magnitude analysis of equations (10a,b):

Consider a weak-field system with characteristic field strength ε and characteristic length-time scale ℓ. Below each term of equations (10a,b), write the order of magnitude of that term:

$$\begin{array}{ccccc} T^{\mu\nu}{}_{,\nu} & = & -\Gamma^{\mu}{}_{\alpha\nu}T^{\alpha\nu} & - & \Gamma^{\nu}{}_{\alpha\nu}T^{\mu\alpha} \\ (T^{00}/\ell) & & (\varepsilon/\ell)(T^{00}) & & (\varepsilon/\ell)(T^{00}) \end{array} \tag{13a}$$

$$\begin{array}{ccccccccc} \Box_s \bar{h}^{\mu\nu} & = & -16\pi(-g)^{1/2}\,T^{\mu\nu} & - & 16\pi(-g)^{1/2}\,t^{\mu\nu}_{L-L} & - & (-g)^{-1/2}\,\bar{h}^{\mu\alpha}{}_{,\beta}\,\bar{h}^{\nu\beta}{}_{,\alpha} \\ (\varepsilon/\ell^2) & & T^{00} & & (\varepsilon^2/\ell^2) & & (\varepsilon^2/\ell^2) \end{array} \tag{13b}$$

Equation (13a) shows that fractional errors ε^{n_h} in $\bar{h}^{\mu\nu}$ produce fractional errors ε^{n_h+1} in $T^{\mu\nu}$; i.e., $\varepsilon^{n_T} \geq \varepsilon^{n_h+1}$; i.e.,

$$n_T \leq n_h + 1 \quad . \tag{14a}$$

Equation (13b)--together with the order-of-magnitude field equation $T^{00} \sim \varepsilon/\ell^2$--shows that fractional errors ε^{n_T} in $T^{\mu\nu}$ produce fractional errors ε^{n_T} in $\bar{h}^{\mu\nu}$; i.e., $\varepsilon^{n_h} \geq \varepsilon^{n_T}$; i.e.,

$$n_h \leq n_T \quad . \tag{14b}$$

Equations (14a,b) can be restated as the following constraints on the order indices of any self-consistent approximation formalism:

$$n_h = n_T - 1 \quad \text{or} \quad n_h = n_T \quad . \tag{15}$$

In other words, <u>the order (n_T, n_h) of any approximation formalism must be either $(n,n-1)$ or (n,n) for some integer n</u>.

Suppose that a specific system has been analyzed using an approximation formalism of order $(n,n-1)$. Denote by ${}_{n}T^{\mu\nu}(x^{\alpha})$ and ${}_{(n-1)}\bar{h}^{\mu\nu}(x^{\alpha})$ the explicit expressions obtained in that analysis for the system's stress-energy tensor and gravitational field. From these expressions it is straightforward to generate an "improved" gravitational field ${}_n\bar{h}^{\mu\nu}(x^\alpha)$ with fractional errors ε^n. The key to doing this is the structure of the field equations (10b): In these field equations, fractional errors of order ε^{n-1} in $\bar{h}^{\mu\nu}$ produce fractional errors of order ε^n in both $\Box_s$ and the expression

$$-16\pi(-g)^{1/2}(T^{\mu\nu} + t^{\mu\nu}_{L-L}) - (-g)^{-1/2}\,\bar{h}^{\mu\alpha}{}_{,\beta}\,\bar{h}^{\nu\beta}{}_{,\alpha} \quad .$$

Hence, ${}_n\bar{h}^{\mu\nu}(t,\underset{\sim}{x})$ satisfies the differential equation

$$_{(n,n-1)}\Box_s\ {}_n\bar{h}^{\mu\nu} = {}_{(n,n-1)}[-16\pi(-g)^{1/2}(T^{\mu\nu} + t^{\mu\nu}_{L-L}) - (-g)^{-1/2}\,\bar{h}^{\mu\alpha}{}_{,\beta}\bar{h}^{\nu\beta}{}_{,\alpha}] \ . \tag{16}$$

Here the prefix (n,n-1) means that a quantity is to be calculated, with fractional error ε^n, using ${}_nT^{\mu\nu}$ and ${}_{(n-1)}\bar{h}^{\mu\nu}$. This inhomogeneous, linear wave equation for ${}_n\bar{h}^{\mu\nu}$ can be solved using the retarded scalar Green's function for curved spacetime (DeWitt and Brehme 1960):

$$_{(n-1)}G(P',P) \equiv \left(\begin{array}{l}\text{the retarded scalar Green's function for the curved} \\ \text{spacetime with the metric } {}_{(n-1)}g_{\mu\nu} \text{ of the (n,n-1)} \\ \text{approximation--a Green's function with fractional} \\ \text{errors } \varepsilon^n\end{array}\right). \tag{17}$$

The result is

$$_n\bar{h}^{\mu\nu}(P) = \int \ {}_{(n,n-1)}[\,16\pi(-g)(T^{\mu\nu} + t^{\mu\nu}_{L-L}) + \bar{h}^{\mu\alpha}{}_{,\beta}\ \bar{h}^{\nu\beta}{}_{,\alpha}]_{P'}\ {}_{(n-1)}G(P',P)\,d^4x' . \tag{18}$$

This paragraph can be summarized as follows: <u>Any approximation formalism of order (n,n-1), when augmented by equation (18) for ${}_n\bar{h}^{\mu\nu}$, becomes an approximation formalism of order (n,n).</u>

Special relativity and linearized theory provide a simple example of the above remarks: Special relativity is the approximation formalism of order (1,0) which one obtains by the extreme truncation process of setting $\bar{h}^{\mu\nu} = 0$ in equations (9) and (10):

$$ {}_0\bar{h}^{\mu\nu} = 0 \quad , \quad {}_0t^{\mu\nu}_{L-L} = 0 \quad , \quad {}_0g_{\mu\nu} = \eta_{\mu\nu} \quad , \quad {}_1T^{\mu\nu}{}_{,\nu} = 0 \quad . \tag{19a} $$

The retarded scalar Green's function for a space with metric ${}_0g_{\mu\nu} = \eta_{\mu\nu}$ is

$$ {}_0G(\mathcal{P}',\mathcal{P}) = (4\pi)^{-1}\, \delta_{ret}[\tfrac{1}{2}\,\eta_{\rho\sigma}(x^\rho - x^{\rho'})(x^\sigma - x^{\sigma'})] \quad . \tag{19b} $$

(Here δ_{ret} is zero if $\mathcal{P}$ lies in the causal past of $\mathcal{P}'$, and it is the Dirac delta function otherwise). Hence, equation (18)--by which one must augment special relativity in order to obtain a formalism of order (1,1)--has the form

$$ \begin{aligned} {}_1\bar{h}^{\mu\nu} &= 4\int {}_1T^{\mu\nu}(\mathcal{P}')\delta_{ret}[\tfrac{1}{2}\,\eta_{\rho\sigma}(x^\rho - x^{\rho'})(x^\sigma - x^{\sigma'})]d^4x' \\ &= 4\int \frac{{}_1T^{\mu\nu}(x^0 - |\underset{\sim}{x}-\underset{\sim}{x}'|, \underset{\sim}{x}')}{|\underset{\sim}{x}-\underset{\sim}{x}'|}\, d^3x' \quad . \end{aligned} \tag{20} $$

The resulting (1,1) formalism [equations (19) augmented by equation (20)] is the "linearized theory of gravity"; [see, e.g., §7.1, Box 7.1, and Chapter 18 of MTW.]

Newtonian theory and the "quadrupole-moment formalism for wave generation" are another example. Newtonian theory is the weak-field formalism of order (2,1) which one obtains by not only truncating each series that appears in equations (9) and (10), but by also imposing the slow-motion and small-stress assumptions

$$v^2 \equiv |T^{0j} \underset{\sim}{e}_j / T^{00}|^2 \lesssim \varepsilon \quad , \qquad |T^{ij}/T^{00}| \lesssim \varepsilon \quad , \tag{21}$$

$$\text{(size of system)/(characteristic time scale of changes)} \lesssim \varepsilon^{1/2} \ .$$

Equation (18), by which one augments Newtonian theory in order to obtain a formalism of order (2,2), has the form, when evaluated in the radiation zone

$$_2\bar{h}^{TT}_{ij}(t,\underset{\sim}{x}) = (2/r)\ \ddot{I\!\!-}^{TT}_{ij}(t-r) = \text{(gravitational radiation field)} \ . \tag{22}$$

Here $I\!\!-_{ij}$ is the reduced quadrupole moment of the source, and TT denotes "transverse-traceless" part. This is the standard wave-generation formula of the quadrupole-moment formalism; see Chapter 36 of MTW.

IV. WAVE GENERATION BY A WEAK-FIELD SYSTEM

a) Motivation

Weak-field systems are of two types: those with negligible self-gravitational forces (rotating laboratory rods; microwave cavities; etc.), and those whose internal motions are significantly influenced by self-gravity (pulsating stars; binary star systems; etc.).

For a system with negligible self-gravity, special relativity gives a fairly accurate description of the internal motions; and, consequently, linearized theory [the (1,1) formalism obtained by attaching equation (18) or (20) onto special relativity] gives a fairly accurate description of gravitational-wave generation.

For most weak-field astrophysical systems, self-gravitational forces are important. In this case, when analyzing a system's internal motions, one must use a formalism of order (2,1); and when calculating the waves those motions generate, one must augment the (2,1) formalism by equation (18), thereby raising its order to (2,2).[3] If the system has

[3] In very special cases second-order gravitational forces may be as important, for the system's motions, as first-order forces. An example is a radially pulsating, weak-field star with adiabatic index very near 4/3 (Chandrasekhar 1964); see also the discussion accompanying equations (61) below. When analyzing such systems one needs formalisms of order (3,2) and (3,3).

slow internal motions and weak internal stresses, Newtonian theory [order (2,1)] will suffice for analyzing its motions, and the quadrupole-moment formalism [order (2,2)] will suffice for wave generation. However, for analyzing fast-motion systems (e.g., two stars flying past each other with high velocity and deflecting each other slightly--the relativistic bremsstrahlung problem), one needs unrestricted (2,1) and (2,2) formalisms. The objective of the next two sections is to derive such formalisms.

b) The Post-Linear Formalism

A weak-field formalism of order (2,1), unrestricted by any constraints on velocities or stresses, can be obtained by truncating equations (9) and (10) at the appropriate order:

$$ {}_1\mathfrak{g}^{\mu\nu} = \eta^{\mu\nu} - {}_1\bar{h}^{\mu\nu} \quad , \tag{23a} $$

$$ (-{}_1g) = 1 - {}_1\bar{h} = 1 + {}_1h \quad , \tag{23b} $$

$$ {}_1g^{\mu\nu} = \eta^{\mu\nu} - {}_1h^{\mu\nu} \quad (\text{where } {}_1h^{\mu\nu} = {}_1\bar{h}^{\mu\nu} - \tfrac{1}{2}\, {}_1\bar{h}\eta^{\mu\nu}) \quad , \tag{23c} $$

$$ {}_1g_{\mu\nu} = \eta_{\mu\nu} + {}_1h_{\mu\nu} \quad , \tag{23d} $$

$$ {}_1\Gamma^{\mu}{}_{\alpha\beta} = \tfrac{1}{2}({}_1h^{\mu}{}_{\alpha,\beta} + {}_1h^{\mu}{}_{\beta,\alpha} - {}_1h_{\alpha\beta}{}^{,\mu}) \quad , \tag{23e} $$

$$ {}_1\Box_s = (\eta^{\alpha\beta} - {}_1h^{\alpha\beta})\, \partial_\alpha \partial_\beta \quad , \tag{23f} $$

$$ \begin{aligned} {}_1t^{\alpha\beta}_{L-L} = (16\pi)^{-1} \{ & \tfrac{1}{2}\eta^{\alpha\beta}\, \eta_{\lambda\mu}\, {}_1\bar{h}^{\lambda\nu}{}_{,\rho}\, {}_1\bar{h}^{\rho\mu}{}_{,\nu} + \eta_{\lambda\mu}\, \eta^{\nu\rho}\, {}_1\bar{h}^{\alpha\lambda}{}_{,\nu}\, {}_1\bar{h}^{\beta\mu}{}_{,\rho} \\ & - (\eta^{\alpha\lambda}\, \eta_{\mu\nu}\, {}_1\bar{h}^{\beta\nu}{}_{,\rho}\, {}_1\bar{h}^{\mu\rho}{}_{,\lambda} + \eta^{\beta\lambda}\, \eta_{\mu\nu}\, {}_1\bar{h}^{\alpha\nu}{}_{,\rho}\, {}_1\bar{h}^{\mu\rho}{}_{,\lambda}) \\ & + \tfrac{1}{8}(2\eta^{\alpha\lambda}\, \eta^{\beta\mu} - \eta^{\alpha\beta}\, \eta^{\lambda\mu})(2\eta_{\nu\rho}\, \eta_{\sigma\tau} - \eta_{\rho\sigma}\, \eta_{\nu\tau})\, {}_1\bar{h}^{\nu\tau}{}_{,\lambda}\, {}_1\bar{h}^{\rho\sigma}{}_{,\mu} \} \quad , \end{aligned} \tag{23g} $$

$$ {}_2T^{\mu\nu}{}_{,\nu} = -{}_1\Gamma^{\mu}{}_{\alpha\nu}\, {}_2T^{\alpha\nu} - {}_1\Gamma^{\nu}{}_{\alpha\nu}\, {}_2T^{\mu\alpha} \quad , \tag{24a} $$

$$ \eta^{\alpha\beta}\, {}_1\bar{h}^{\mu\nu}{}_{,\alpha\beta} = -16\pi\, {}_2T^{\mu\nu} \quad . \tag{24b} $$

We shall refer to the formalism described by these equations as the "post-linear formalism". To analyze a system using the post-linear formalism, one must first specify the functional dependence of the stress-energy tensor ${}_2T^{\mu\nu}$ on the system's nongravitational variables (e.g., density, pressure, velocities, electromagnetic field tensor, $\cdots$) and on the gravitational field ${}_1\bar{h}^{\mu\nu}$; and one must then solve equations (24a,b) simultaneously for the system's motions (${}_2T^{\mu\nu}$ accurate up to fractional errors $\sim \varepsilon^2$) and for the gravitational field (${}_1\bar{h}^{\mu\nu}$ accurate up to fractional errors $\sim \varepsilon$). Paper III will carry out such a calculation for the motion of two stars of arbitrary relative masses and velocities, which fly past each other with large impact parameter.

c) The Post-Linear Wave-Generation Formalism

Having calculated a system's internal structure and motions using the post-linear formalism, one can then calculate the gravitational waves the system emits, ${}_2\bar{h}^{\mu\nu}$, by evaluating expression (18). In evaluating (18) one needs an explicit expression for the retarded Green's function ${}_1G(P',P)$ associated with the metric ${}_1g_{\mu\nu} = \eta_{\mu\nu} + {}_1h_{\mu\nu}$. In the next subsection (§i) we derive ${}_1G(P',P)$; then in §ii we place constraints on our system which simplify ${}_1G(P',P)$; and finally in §iii we use ${}_1G(P',P)$ to evaluate the wave field ${}_2\bar{h}^{\mu\nu}$.

i. The Green's Function ${}_1G(P',P)$

We shall obtain ${}_1G(P',P)$ by taking the weak-field limit of the exact Green's function $G(P',P)$ for a space described by an exact metric $g_{\mu\nu}$. The exact Green's function is formally rather simple, so long as the congruence of geodesics that emanate from the source point P' does not get focussed so strongly along the future light cone of P' that geodesics cross. Henceforth we shall assume "no crossing of geodesics on the light cone." Later [eqs. (48), (48'), (48") below] we shall examine the constraints placed on the radiating system by this "no-crossing" assumption.

DeWitt and Brehme (1960) have derived the exact Green's function $G(P',P)$ for the case of no crossing. Their Green's function consists of a "direct part" and a "tail"

$$G(P',P) = G^{\text{direct}} + G^{\text{tail}} \quad . \tag{25}$$

The direct part is nonzero only if P lies on the future light cone of P' [denoted $\dot{J}^+(P')$]. By virtue of the "no-crossing" assumption, when P is near $\dot{J}^+(P')$ there is a unique geodesic from P' to P with a unique squared length

$$\Omega(P',P) \equiv \begin{pmatrix}\text{"World function",} \\ \text{see Synge (1960)}\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-1 \text{ for timelike geodesic} \\ +1 \text{ for spacelike geodesic}\end{pmatrix}\begin{pmatrix}\text{proper distance} \\ \text{along geodesic}\end{pmatrix}^2 \tag{26}$$

$= \sigma$ in notation of DeWitt and Brehme (1960).

Because $\dot{J}^+(P')$ is characterized by $\Omega = 0$, G^{direct} must have the form

$$G^{\text{direct}}(P',P) = (4\pi)^{-1}[\Delta(P',P)]^{1/2}\,\delta_{\text{ret}}[\Omega(P',P)] \quad , \tag{27}$$

where δ_{ret} is the Dirac delta function on and near $\dot{J}^+(P')$, and is zero on and near the past light cone $[\dot{J}^-(P')]$. The quantity $\Delta(P',P)$ is an amplitude factor which would be unity in flat spacetime, but in curved spacetime is given by

$$\Delta(P',P) = \frac{\det\|\partial^2\Omega/\partial x^\alpha \partial x^{\beta'}\|}{|g(P)\ g(P')|^{1/2}} \quad . \tag{28}$$

We shall use an expression for the tail different from, but equivalent to that given by DeWitt and Brehme. To derive our expression we insert equations (25) and (27) into the wave equation

$$\Box_s G(P',P) = -[g(P)g(P')]^{-1/4}\delta(x^0-x^{0'})\ \delta(x^1-x^{1'})\ \delta(x^2-x^{2'})\ \delta(x^3-x^{3'}) \ . \tag{29}$$

The result is

$$\Box_s G^{\text{tail}} = -(4\pi)^{-1}\ \{(\Box_s\Delta^{1/2})\ \delta(\Omega) + [2\nabla_K\Delta^{1/2} + (\Box_s\Omega)\Delta^{1/2}]\ \delta'(\Omega) + (\nabla\Omega)^2\ \Delta^{1/2}\ \delta''(\Omega)\} \quad , \tag{30a}$$

where δ' and δ'' are the first and second derivatives of the retarded Dirac delta function, ∇ is the 4-dimensional gradient operator, and ∇_K is covariant derivative along the 4-vector

$$K \equiv \nabla\Omega \quad . \tag{30b}$$

[Here and below we suppress the subscript "ret" on $\delta(\Omega)$.] We then manipulate expression (30a) using the relations

$$\Omega\delta''(\Omega) = -2\delta'(\Omega) \ , \quad (\nabla\Omega)^2 = 2\Omega \ , \quad \Box_s\Omega - 4 = -\Delta^{-1}\,\nabla_K\Delta \quad . \tag{30c}$$

[The first of these is a standard identity for Dirac delta functions; the second and third are eqs. (1.11) and (1.63) of DeWitt and Brehme (1960)]. The result is

$$\Box_s\, G^{\mathrm{tail}} = -(4\pi)^{-1}(\Box_s\,\Delta^{1/2})\;\delta(\Omega) \quad . \tag{31}$$

We then use relations (30c) and the relation $\Delta(P\,,P) = 1$ to rewrite this in the form

$$\Box_s[G^{\mathrm{tail}} - (4\pi)^{-1}\,(1-\Delta^{1/2})\;\delta(\Omega)] = +(4\pi)^{-1}(\nabla_K\,\ell n\,\Delta)\delta'(\Omega) \ . \tag{32}$$

Equation (31) tells us that G^{tail} jumps from zero outside the light cone to a finite value inside the cone, without having any singularities on the cone. Equation (32) allows us to write (restoring the subscript "ret")

$$G^{\mathrm{tail}}(P',P) = \begin{cases} 0 \qquad \text{if} \quad P' \notin I^-(P) & \text{(33a)} \\ -(4\pi)^{-1}\displaystyle\int [\ell n\,\Delta(P',P'')]_{,\alpha''}\,[\Omega(P',P'')]^{,\alpha''}\;\delta'_{\mathrm{ret}}[\Omega(P',P'')]\,\times & \\ \qquad \times\; G(P'',P)[-g(P'')]^{1/2}\,d^4x'' \qquad \text{if} \quad P' \in I^-(P) \quad . & \text{(33b)} \end{cases}$$

Here $I^-(P)$ means "the interior of the past light cone of P;" and condition (33a) suppresses the unwanted light-cone part of (33b) [i.e., suppresses $(4\pi)^{-1}\,(1-\Delta^{1/2})\;\delta(\Omega)$].

Equation (33) is the form of the tail which we shall use. This form was suggested to us by the work of Peters (1966).

We now specialize the above equations for the retarded Green's function to the case of a weak gravitational field ${}_1g_{\mu\nu} = \eta_{\mu\nu} + {}_1h_{\mu\nu}$, beginning with equation (26) for the world function. Let λ be an affine parameter along the geodesic linking P' to P

$$\mathcal{C}(\lambda) \equiv \text{geodesic with coordinates } \xi^\alpha(\lambda) \; ;$$

$$\mathcal{C}(0) = P'\Big| \; , \qquad \mathcal{C}(1) = P\Big| \; , \qquad 0 \le \lambda \le 1 \quad . \tag{34}$$

Then equation (26) can be rewritten in the form [cf. Synge (1960), p. 47]

$$\Omega(P',P) = \int_{\mathcal{C}} \frac{1}{2} \, g_{\mu\nu} (d\xi^\mu/d\lambda)(d\xi^\nu/d\lambda) \; d\lambda \quad . \tag{35}$$

The right-hand side is actually an action principle for the geodesic equation [cf. MTW, Box 13.3]. Therefore, if we evaluate the integral along the "straight line"

$${}_0\mathcal{C}(\lambda): \qquad \xi^\alpha(\lambda) \equiv x^{\alpha'} + \lambda(x^\alpha - x^{\alpha'}) \tag{36}$$

(see Fig. 1), which differs by a fractional amount of $O(\varepsilon)$ from the true geodesic $\mathcal{C}(\lambda)$, we will make fractional errors in Ω of $O(\varepsilon^2)$. Such errors are acceptable in ${}_1G(P',P)$, since its frational errors are also $O(\varepsilon^2)$; cf., eq. (17). The result of integrating expression (35) along the slightly wrong curve ${}_0\mathcal{C}(\lambda)$ is

$${}_1\Omega(P',P) = {}_0\Omega(P',P) + \gamma(P',P) \quad , \tag{37}$$

where

$${}_0\Omega(P',P) \equiv \frac{1}{2} \, X^\alpha X^\beta \, \eta_{\alpha\beta} \quad , \tag{38a}$$

$$\gamma(P',P) \equiv \frac{1}{2} X^\alpha X^\beta \int_0^{\mathcal{C}} {}_1h_{\alpha\beta} d\lambda \quad , \tag{38b}$$

$$X^\alpha \equiv x^\alpha - x^{\alpha'} \quad . \tag{38c}$$

Equation (37) is the desired expression for the world function. Turn next to the amplitude factor $\Delta(P',P)$. Either by direct calculation from eqs. (28), (23b), (37), (38), and (B12), or by invoking eq. (95) on page 63 of Synge (1960), one arrives at the expression

$$
\begin{aligned}
{}_1\Delta(P',P) &\equiv \frac{\det \| {}_1\Omega_{,\alpha\beta'} \|}{|{}_1g\ {}_1g'|^{1/2}} = (1 - \tfrac{1}{2}\ {}_1h - \tfrac{1}{2}\ {}_1h') \det \| {}_1\Omega_{,\alpha\beta'} \| \\
&= 1 + 2\alpha(P',P) \quad .
\end{aligned} \tag{39}
$$

Here

$$\alpha(P',P) \equiv \frac{1}{2} X^\alpha X^\beta \int_0^{\mathcal{C}} {}_1R_{\alpha\beta}\ \lambda(1-\lambda)\ d\lambda \quad , \tag{40a}$$

where ${}_1R_{\alpha\beta}$ is the Ricci tensor, accurate to first order in ${}_1h_{\mu\nu}$:

$${}_1R_{\alpha\beta} = -\frac{1}{2}\ {}_1h_{\alpha\beta,\rho}{}^{\rho} \quad . \tag{40b}$$

In eq. (39) we have simplified notation by using a prime to denote quantities evaluated at P', i.e., ${}_1h' \equiv {}_1h(P')$ while ${}_1h \equiv {}_1h(P)$. Henceforth we shall reserve primes for this purpose--except that δ' and δ'' are still derivatives of Dirac delta functions.

Turn next to the "source term" $(\ell n\, \Delta)_{,\alpha''}\, \Omega^{,\alpha''}$ for the tail (eq. 33). The tail itself is of $O(\varepsilon)$ compared to the direct part of the Green's function; therefore we can permit fractional errors of $O(\varepsilon)$ in the tail--which means we can use the zero-order value of $\Omega^{,\alpha''}$ in the source of the tail:

$$[{}_0\Omega(P',P'')]^{,\alpha''} = x^{\alpha''} - x^{\alpha'} \equiv X^{\alpha''} \quad . \tag{41}$$

By combining this with equation (39) and by using equation (B7) of Appendix B, we bring the source of the tail into the form

$$[\ln {}_1\Delta(P',P'')]_{,\alpha''} [{}_0\Omega(P',P'')]^{,\alpha''} = \beta(P',P'') \quad , \tag{42}$$

where

$$\beta(P',P'') \equiv X^{\alpha''}X^{\beta''} \int_{{}_0\mathcal{C}''} {}_1R_{\alpha\beta}\, \lambda^2 \, d\lambda \tag{43a}$$

and ${}_0\mathcal{C}''$ is the "straight line" from P' to P'' (see Fig. 1)

$${}_0\mathcal{C}'' : \quad \xi^{\alpha} = x^{\alpha'} + \lambda X^{\alpha''} \quad . \tag{43b}$$

Turn, finally, to the propagator $G(P'',P)$ and the volume element $(-g'')^{1/2}\, d^4x''$ which appear in equation (33) for the tail. Because the tail is of $O(\varepsilon)$ compared to the direct part of the Green's function, we can ignore all curved space corrections in the amplitude of the propagator (but not its phase), and in the volume element:

$$G(P'',P)(-g'')^{1/2}\, d^4x'' = (4\pi)^{-1}\, \delta_{ret}[{}_1\Omega(P'',P)]\, d^4x'' \tag{44}$$

$$\text{in expression (33b) for } {}_1G^{tail} \quad .$$

All of the pieces for the first-order Green's function are now at hand. By combining them [eqs. (25), (27), (33), (39), (42), and (44)] we obtain the following result:

$${}_1G(P',P) = {}_1G^{direct} + {}_1G^{tail} \quad , \tag{45a}$$

$$ {}_1G^{\text{direct}} = (4\pi)^{-1}\,[1+\alpha(P',P)]\;\delta_{\text{ret}}[{}_1\Omega(P',P)] \quad , \tag{45b} $$

$$ {}_1G^{\text{tail}} = \begin{cases} 0 & \text{if} \quad P' \notin I^-(P) \\ -(4\pi)^{-2} \int \beta(P',P'')\;\delta'_{\text{ret}}[{}_1\Omega(P',P'')]\;\delta_{\text{ret}}[{}_1\Omega(P'',P)]\; d^4x'' & \\ \qquad \text{if} \quad P' \in I^-(P) \quad . \end{cases} \tag{45c} $$

Here ${}_1\Omega(P',P)$ [and similarly ${}_1\Omega(P',P'')$ and ${}_1\Omega(P'',P)$] is given by equations (37) and (38), $\alpha(P',P)$ (the "focussing function") is defined by expressions (40), and $\beta(P',P'')$ (the "tail generator") is defined by expressions (43).

ii. Constraints Designed to Simplify the Green's Function

Expression (45) for the Green's function is valid only if geodesics emanating from P' fail to cross on and near $\dot{J}^+(P')$. Crossing would be caused by gravitational focussing; and at any crossing point, the exact amplitude factor $\Delta(P',P)$ would diverge. Thus, the criterion for no crossing is finiteness of Δ along $\dot{J}^+(P')$.

Consider our first-order expression (39), (40) for ${}_1\Delta$. Evaluate it in the mean rest-frame of the source, with coordinates centered on P' so that

$$ \begin{aligned} &x^\alpha = rn^\alpha \;, \quad \overline{n^0} \equiv 1 \;, \quad \underset{\sim}{n} = (\text{unit spatial vector pointing from } P' \text{ to } P) \;, \\ &r = (\text{spatial distance from } P' \text{ to } P) \;, \\ &\lambda = \bar{r}/r \equiv (\text{fractional distance from } P' \text{ to } P) \;; \end{aligned} \tag{46} $$

and invoke the first-order field equation ${}_1R_{\alpha\beta} = ({}_1T_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\,{}_1T)$. The result is

$$ {}_1\Delta(P',P) = 1 + 2\alpha \quad , $$

$$ \alpha = \frac{1}{2}\int_0^r (n^\alpha n^\beta \; {}_1T_{\alpha\beta}) \; \bar{r}(1-\bar{r}/r) \; d\bar{r} \quad . \tag{47} $$

This expression for ${}_1\Delta$ can never diverge if the source is bounded, because once the integration point $\bar{r}$ gets outside the source, ${}_1T_{\alpha\beta}$ vanishes and ${}_1\Delta$ stops increasing. However, if the focussing function α approaches unity inside the source, then second-order and higher effects will come into play. As one moves out into the vacuum beyond the source, those second-order effects will be essentially those of the "focussing" or "Raychaudhuri" equation; they will produce a divergence. Thus, the constraint

$$ \text{CONSTRAINT:} \quad \alpha(P',P) \ll 1 \;\; \text{for all} \;\; P' \;\; \text{and} \;\; P \tag{48} $$

is necessary for the validity of the first-order analysis, and simultaneously protects us from "geodesic crossing."

For a system that is roughly homogeneous with mass M and linear size L, equation (47) gives

$$ \alpha \sim (M/L) \sim \varepsilon \ll 1 \quad ; \tag{48'} $$

so there is no problem in satisfying the constraint (48). However, for a highly inhomogeneous sytem (lumps of mass m and size ℓ, separated by distances $b \gg \ell$), and for rays originating in one lump and passing through another, equation (47) gives

$$ \alpha \sim b(m/\ell^3)\ell \sim (b/\ell)(m/\ell) \quad . $$

In this case the constraint (48) is significant: it says that to avoid too much ray focussing, the lumps must not be too far apart

$$(b/\ell) << (\ell/m) \sim 10^6 (\ell/R_\odot)(M_\odot/m) \quad . \tag{48''}$$

The Green's function (45) would be much easier to use if, throughout it, we could replace the first-order world function ${}_1\Omega$ by its zero-order approximation ${}_0\Omega = \frac{1}{2}\eta_{\mu\nu} X^\mu X^\nu$. Let us examine ${}_1\Omega$ [eqs. (37) and (38)] in the rest frame of our source, for points P on or near $\dot{J}_+(P')$:

$$\begin{aligned} {}_1\Omega(P',P) &= \frac{1}{2}(X^0+X)[-X^0 + X + 2\gamma(P',P)/(X^0+X)] \\ &\cong X(-X^0+X+\Delta t_S) \end{aligned} \tag{49a}$$

where

$$X \equiv |\underset{\sim}{X}| = (\text{distance from source to field point}) \tag{49b}$$

$$\Delta t_S \equiv \gamma(P',P)/X = (\text{"Shapiro time delay"}) \quad . \tag{49c}$$

For field points P far outside the source, the dominant contribution to the Shapiro time delay is the asymptotic "1/r" field of the source. It produces a huge delay of

$$\Lambda \equiv 2M \,\ell n(X/L) = \begin{bmatrix}\text{Shapiro time delay due to} \\ \text{asymptotic field of source}\end{bmatrix}$$

$$M \equiv (\text{mass of source}) \tag{50}$$

$$L \equiv (\text{characteristic size of source}) \quad .$$

This delay is time independent and is independent of where inside the source P' is located (aside from a negligible piece of size $\sim 2ML/X$); therefore its only effect on the radiation is to delay the arrival time at a given radius. Henceforth, for ease of calculation, we shall remove this constant delay from the argument of our Green's function. We can always reinsert it at the end of the calculation if we wish. With this constant delay removed, we can rewrite ${}_1\Omega$ as

$$
\begin{aligned}
{}_1\Omega(\mathcal{P}',\mathcal{P}) &\mathrel{\overset{\sim}{=}} X[-X^0 + X + (\Delta t_S - \Lambda)_{\mathcal{P}'\mathcal{P}}] \\
&\mathrel{\overset{\sim}{=}} {}_0\Omega(\mathcal{P}',\mathcal{P}) + [\gamma(\mathcal{P}',\mathcal{P}) + \Lambda X^\mu U_\mu] \qquad (51)
\end{aligned}
$$

for $\mathcal{P}'$ inside the source and $\mathcal{P}$ far outside it ,

where

$$U_\mu = P_\mu/M = \text{4-velocity of source;} \quad X^\mu U_\mu = -X \quad . \qquad (52)$$

The remaining "internally-produced" delay between $\mathcal{P}'$ and $\mathcal{P}$, $\Delta t_S - \Lambda$, is of the same order of magnitude as the total delay between two internal points $\mathcal{P}'$ and $\mathcal{P}''$:

$$
\begin{aligned}
(\Delta t_S - \Lambda)_{\mathcal{P}'\mathcal{P}} \sim (\Delta t_S - \Lambda)_{\mathcal{P}''\mathcal{P}} \sim (\Delta t_S)_{\mathcal{P}'\mathcal{P}''} &\sim \int_{\text{across source}} {}_1h_{00}\, d\bar{r} \\
&\sim M \quad \text{for homogeneous source} \qquad (53) \\
&\sim m\, \ell n(b/\ell) \text{ for lumpy source} \quad .
\end{aligned}
$$

Henceforth we shall <u>assume</u> that this internal time delay is small compared to the characteristic timescale on which the source changes---i.e., small compared to the characteristic reduced wavelength $\lambda\!\!\!^{-}$ of the radiation emitted,

$$\text{CONSTRAINT:} \quad (\Delta t_S)_{\text{internal}} \sim m\, \ell n(b/\ell) \ll \lambda\!\!\!^{-} \quad . \qquad (54)$$

[Example: If $\lambda\!\!\!^{-}$ is 100 times larger than the Schwarzschild radius, 2m, of a lump, then b/ℓ can be as large as $\exp(10) \sim 2 \times 10^4$ without causing problems. Another example: If $\lambda \gtrsim b$ (which is the case for bremsstrahlung), and if $\ell >> m$ (which is required for fields to be weak), then the condition $b \gtrsim \ell$ (separation of lumps bigger than size of lumps) guarantees that constraint (54) is satisfied.]

The constraint (54) allows us to expand our delta functions $\delta({}_1\Omega)$ in powers of the internal time delay. Discarding terms that are quadratic and higher-order in $(\Delta t_S)_{\text{internal}}/\lambda$, we obtain for the Green's function (45)

$$ {}_1G(P',P) = {}_1G^{\text{direct}} + {}_1G^{\text{tail}} \quad , \tag{55a} $$

$$ {}_1G^{\text{direct}}(P',P) = (4\pi)^{-1} \{\delta_{\text{ret}}(\tfrac{1}{2} X^\alpha X^\beta \eta_{\alpha\beta}) + \alpha(P',P)\, \delta_{\text{ret}}(\tfrac{1}{2} X^\alpha X^\beta \eta_{\alpha\beta}) + [\gamma(P',P) + \Lambda X^\mu U_\mu]\, \delta'_{\text{ret}}(\tfrac{1}{2} X^\alpha X^\beta \eta_{\alpha\beta})\} \quad ; \tag{55b} $$

$$ {}_1G^{\text{tail}}(P',P) = \begin{cases} 0 & \text{if} \quad P' \notin I^-(P) \\ -(4\pi)^{-2} \int \beta(P',P'')\, \delta'_{\text{ret}}(\tfrac{1}{2} X^{\alpha''} X^{\beta''} \eta_{\alpha\beta})\, \delta_{\text{ret}}(\tfrac{1}{2} \overline{X}^\alpha \overline{X}^\beta \eta_{\alpha\beta}) d^4x'' & \\ & \text{if} \quad P \in I^-(P) \quad . \end{cases} $$

In these equations

$$ X^\alpha \equiv x^\alpha - x^{\alpha'} \quad , \quad \overline{X}^\alpha \equiv x^\alpha - x^{\alpha''} \quad , \quad X^{\alpha''} \equiv x^{\alpha''} - x^{\alpha'} \quad ; \tag{56} $$

see figure 1.

Equations (55) are our final form for the scalar Green's function in a space with linearized metric ${}_1g_{\mu\nu} = \eta_{\mu\nu} + {}_1h_{\mu\nu}$. This Green's function has fractional errors

$$ |(\text{errors in } {}_1G)/{}_1G| \sim \text{Maximum of } \{\varepsilon^2, \alpha\varepsilon, [(\Delta t_S)_{\text{internal}}/\lambda]\varepsilon\} \text{ in general,} $$
$$ \sim \varepsilon^2 \quad \text{for most sources} \quad ; \tag{57} $$

and it has been stripped of its asymptotic time delay (eq. 50).

iii. The Gravitational-Wave Field ${}_2\overline{h}^{\mu\nu}$

By inserting expressions (55) for ${}_1G(P',P)$ into equation (18) we obtain the following expression for the gravitational field far outside a weak-field source:

$$ {}_2\overline{h}^{\mu\nu} = {}_2\overline{h}^{\mu\nu}_{D} + {}_2\overline{h}^{\mu\nu}_{F} + {}_2\overline{h}^{\mu\nu}_{TR} + {}_2\overline{h}^{\mu\nu}_{W} + {}_2\overline{h}^{\mu\nu}_{TL} \quad ; \tag{58a} $$

$$ {}_2\overline{h}^{\mu\nu}_{D} = 4 \int \delta_{ret}(\tfrac{1}{2}X^\alpha X^\beta \eta_{\alpha\beta})\ {}_2T^{\mu\nu}(P')\ [1 - {}_1\overline{h}(P')]\ d^4x' \quad , \tag{58b} $$

$$ {}_2\overline{h}^{\mu\nu}_{F} = 4 \int \alpha(P',P)\ \delta_{ret}(\tfrac{1}{2}X^\alpha X^\beta \eta_{\alpha\beta})\ {}_2T^{\mu\nu}(P')\ d^4x' \quad , \tag{58c} $$

$$ {}_2\overline{h}^{\mu\nu}_{TR} = 4 \int [\gamma(P',P) + \Lambda X^\alpha U_\alpha]\ \delta'_{ret}(\tfrac{1}{2}X^\alpha X^\beta \eta_{\alpha\beta})\ {}_2T^{\mu\nu}(P')\ d^4x' \quad , \tag{58d} $$

$$ {}_2\overline{h}^{\mu\nu}_{W} = 4 \int \delta_{ret}(\tfrac{1}{2}X^\alpha X^\beta \eta_{\alpha\beta})\ [{}_1t^{\mu\nu}_{L-L} + (16\pi)^{-1}\ {}_1\overline{h}^{\mu\rho}{}_{,\sigma}\ {}_1\overline{h}^{\nu\sigma}{}_{,\rho}]_{\text{at } P'}\ d^4x' , \tag{58e} $$

$$ {}_2\overline{h}^{\mu\nu}_{TL} = (-1/\pi) \iint_{P' \in I^-(P)} \beta(P',P'')\ \delta'_{ret}(\tfrac{1}{2}X^{\alpha''} X^{\beta''} \eta_{\alpha\beta})\ \delta_{ret}(\tfrac{1}{2}\overline{X}^\alpha \overline{X}^\beta \eta_{\alpha\beta}) \times {}_2T^{\mu\nu}(P')\ d^4x''\ d^4x' \quad . \tag{58f} $$

Here P' and P'' are source points with coordinates $x^{\alpha'}$ and $x^{\alpha''}$ (cf.fig.1); the field point P has coordinates x^α; ${}_2T^{\mu\nu}$, ${}_1t^{\mu\nu}_{L-L}$, and ${}_1\overline{h}^{\mu\nu}$ are the stress-energy tensor, the pseudotensor, and the gravitational field obtained by a post-linear analysis [eqs. (23) and (24)]; δ_{ret} is the Dirac delta function on the future light cone of the source and zero on the past light cone; X^α, $\overline{X}^\alpha$, and $X^{\alpha''}$ are

$$ X^\alpha \equiv x^\alpha - x^{\alpha'}, \quad \overline{X}^\alpha \equiv x^\alpha - x^{\alpha''}, \quad X^{\alpha''} \equiv x^{\alpha''} - x^{\alpha'} \quad ; \tag{59a} $$

α, β, and γ are defined by integrals of the first-order Ricci tensor ${}_1R_{\mu\nu}$ and of the metric perturbation ${}_1h_{\mu\nu}$ along the straight line between two points

$$\alpha(P',P) = \frac{1}{2}X^{\alpha}X^{\beta}\int_0^1 {}_1R_{\alpha\beta}(x^{\mu'}+\lambda X^{\mu})\ \lambda(1-\lambda)\ d\lambda \quad , \tag{59b}$$

$$\beta(P',P'') = X^{\alpha''}X^{\beta''}\int_0^1 {}_1R_{\alpha\beta}(x^{\mu'}+\lambda X^{\mu''})\ \lambda^2\ d\lambda \quad , \tag{59c}$$

$$\gamma(P',P) = \frac{1}{2}\ X^{\alpha}X^{\beta}\int_0^1 {}_1h_{\alpha\beta}(x^{\mu'}+\lambda X^{\mu})\ d\lambda \quad ; \tag{59d}$$

$-\Lambda X^{\alpha}U_{\alpha}$ is that portion of γ which is produced by the asymptotic, $1/r$, external field of the source

$$-\Lambda X^{\alpha}U_{\alpha} = \Lambda(-X^{\alpha}U_{\alpha}) = \begin{pmatrix}\text{Shapiro time delay}\\ \text{produced outside source}\end{pmatrix} \times \begin{pmatrix}\text{distance from source}\\ \text{point to field point}\end{pmatrix} \tag{59e}$$

(see §ii above); and $P' \in I^-(P)$ means that the integration (58f) is performed over field points P' that lie inside but not on the past light cone of P.

Each piece of the distant gravitational field ${}_2\bar{h}^{\mu\nu}$ has its own physical origin and significance:

$2\bar{h}^{\mu\nu}_D$ is the "direct field." It is produced by the stress-energy ${}_2T^{\mu\nu}$ and propagates as though spacetime were flat. It includes the zero-order, non-radiative, "$1/r$" field of the source, and also that portion of the radiation produced "directly" by the source's motions. If the internal gravity of the source has negligible influence on the source's structure and evolution, then all other parts of ${}_2\bar{h}^{\mu\nu}$ will be negligible compared to the direct field ("linearized theory"; cf. eq. (20) and the associated discussion).

$2\bar{h}^{\mu\nu}_F$ is the "focussing field". It is the amount by which the direct field is augmented due to focussing as it passes through regions of nonzero Ricci curvature (nonzero stress-energy).

${}_2\bar{h}^{\mu\nu}_{TR}$ is the "transition field" [first discovered in the equations of general relativity by Chitre, Price, and Sandberg (1973,1975); analogue of "electromagnetic transition radiation", Ginzburg and Frank (1946)] . It is the amount by which the direct field changes due to Shapiro-type time delays within the time-varying source.

${}_2\bar{h}^{\mu\nu}_{W}$ is the "whump field." It is the field generated by "gravitational stresses" ${}_1t^{\mu\nu}_{L-L} + (16\pi)^{-1}\ {}_1\bar{h}^{\mu\rho}{}_{,\sigma}\ {}_1\bar{h}^{\nu\sigma}{}_{,\rho}$. We have given it the name "whump" because in our minds we have a heuristic image of gravitational stresses linking various pieces of the source, and going "whumpity-whump-whump" as the source contorts and gyrates.

${}_2\bar{h}^{\mu\nu}_{TL}$ is the "tail field". It is generated by the direct field in those regions where focussing has deformed the geometry of the direct wave fronts.

Although it is useful, heuristically and in calculations, to split ${}_2\bar{h}^{\mu\nu}$ into these five pieces, one should not attribute too much physical significance to each individual piece. For example, no individual piece satisfies the Einstein field equations or the deDonder gauge condition. However, the five individual pieces combine in such a way that their sum does satisfy the field equations and gauge condition; see Appendix C.

V. SLOW-MOTION LIMIT OF THE WAVE-GENERATION FORMULAS

Consider a weak-field system which has slow internal motions and weak internal stresses. Characterize it by the following parameters:

$$
\begin{aligned}
L &\equiv \text{(size of system)} \\
\lambda\!\!\!{}^{-} &\equiv \text{(characteristic time-scale of system)} = \text{(reduced wavelength of radiation)} \\
M &\equiv \text{(mass of system)} \\
v &\equiv (|T^{0j}|/T^{00})_{max} = \text{(maximum internal velocity)} \\
S^2 &\equiv (|T^{ij}|/T^{00})_{max} = \text{maximum of (stress)/(density)} \quad .
\end{aligned}
\tag{60}
$$

Chapter 36 of MTW derives the quadrupole-moment formalism for gravitational wave generation under the following assumptions (eqs. 36.18 of MTW)

$$L/\lambda\!\!\!{}^{-} \ll 1 \quad \text{which implies} \quad v \ll 1 \quad ; \tag{61a}$$

$$M/L \ll L/\lambda\!\!\!{}^{-} \quad , \quad S^2 \ll L/\lambda\!\!\!{}^{-} \quad . \tag{61b}$$

Constraint (61a) is the standard slow-motion assumption--the only assumption truly necessary for validity of the quadrupole-moment formalism (see

Paper II). Constraints (61b) say that the motion must not be <u>too slow</u> if a weak-field calculation is to yield the quadrupole-moment formalism. In terms of the characteristic frequency $\omega \equiv 1/\lambda\!\!\!^{-}$ this "not too slow" assumption says

$$\omega^2 >> (M/L)(M/L^3) \ , \quad \omega^2 >> S^2(S/L)^2 \ . \tag{61b'}$$

A violation of these assumptions occurs, in dynamical systems, only when the gravitational and stress forces counterbalance each other so precisely that second-order gravity, ${}_2\bar{h}^{\mu\nu}$, can affect the motion significantly [cf. Chandrasekhar (1964)]. In this case an analysis based on the post-linear approximation cannot possibly give a correct description of the radiation.

It is instructive to see how the post-linear radiation formulas (58) of this paper yield the quadrupole-moment formalism, when applied to a system satisfying constraints (61).

We begin by combining the direct and whump fields (58b,e) and then breaking them up again, differently

$${}_2\bar{h}^{\mu\nu}_D + {}_2\bar{h}^{\mu\nu}_W = {}_2\bar{h}^{\mu\nu}_{DW1} + {}_2\bar{h}^{\mu\nu}_{DW2} \qquad ; \tag{62a}$$

$${}_2\bar{h}^{\mu\nu}_{DW1} \equiv 4 \int \delta_{ret}(\tfrac{1}{2}X^\alpha X^\beta \ \eta_{\alpha\beta})[(-{}_1g)({}_2T^{\mu\nu} + {}_1t^{\mu\nu}_{L-L})]_{at\ P'} \ d^4x' \ , \tag{62b}$$

$${}_2\bar{h}^{\mu\nu}_{DW2} \equiv (1/4\pi) \int \delta_{ret}(\tfrac{1}{2}X^\alpha X^\beta \ \eta_{\alpha\beta})[{}_1\bar{h}^{\mu\rho}{}_{,\sigma}\ {}_1\bar{h}^{\nu\sigma}{}_{,\rho}]_{at\ P'} \ d^4x' \ . \tag{62c}$$

We then evaluate expression (62b) in the rest frame of the source

$${}_2\bar{h}^{\mu\nu}_{DW1} = 4 \int \frac{(-{}_1g)({}_2T^{\mu\nu} + {}_1t^{\mu\nu}_{L-L})_{ret}}{|\underset{\sim}{x} - \underset{\sim}{x}'|} \ d^3x' \ ; \tag{63}$$

and by carrying out the analysis of[4] MTW §36.10, we bring the spatial

[4]Note that $(-_1g)({}_2T^{\mu\nu} + {}_1t^{\mu\nu}_{L-L})$ here plays the same role as $T^{\mu\nu} + t^{\mu\nu}$ in MTW. The key properties which they share are (i) vanishing coordinate divergence; (ii) same role in retarded integral for ${}_2\bar{h}^{\mu\nu}$.

transverse-traceless part of this field into the form

$$[{}_2h^{jk}_{DW1}(t,\underset{\sim}{x})]^{TT} = (2/r)(d^2/dt^2)\ \text{\ss}^{TT}_{jk}(t-r) \sim (M/r)(L/\lambda\!\!\!^{-})^2 \quad . \tag{64}$$

Here $\text{\ss}_{jk}$ is the "reduced quadrupole moment" of the source, and $\text{\ss}^{TT}_{jk}$ is its transverse traceless part. This is the standard quadrupole-moment formula for the radiation field.

An order-of-magnitude analysis shows that all other parts of our expression (58) for ${}_2\bar{h}_{jk}$ are negligible. In particular, by using the following relations valid for the source's interior

$$ {}_1\bar{h}^{00} \sim M/L\ , \quad {}_1\bar{h}^{0j} \sim Mv/L\ , \quad {}_1\bar{h}^{jk} \sim MS^2/L, \quad T^{jk} = MS^2/L^3\ ,$$

$$ {}_1\bar{h}^{\alpha\beta}{}_{,0} \sim {}_1\bar{h}^{\alpha\beta}/\lambda\!\!\!^{-}, \quad {}_1\bar{h}^{\alpha\beta}{}_{,j} \sim {}_1\bar{h}^{\alpha\beta}/L\ , \quad \beta \sim M/L\ , \tag{65a}$$

as well as the relations

$$\alpha \sim M/L\ , \quad (\gamma + \Lambda X^{\alpha}U_{\alpha}) \sim rM \quad , \tag{65b}$$

we obtain for the ratio of each other part to the "DW1" part (eq. 64):

$$|{}_2\bar{h}^{jk}_{DW2}/({}_2\bar{h}^{jk}_{DW1})^{TT}| \sim (\tfrac{M}{L})(v + \tfrac{S^2}{L/\lambda\!\!\!^{-}})^2 \ll 1 \quad , \tag{66a}$$

$$|{}_2\bar{h}^{jk}_{F}/({}_2\bar{h}^{jk}_{DW1})^{TT}| \sim (\tfrac{M/L}{L/\lambda\!\!\!^{-}})(\tfrac{S^2}{L/\lambda\!\!\!^{-}}) \ll 1 \quad , \tag{66b}$$

$$\left|{}_2\bar{h}^{jk}_{TR}/({}_2\bar{h}^{jk}_{DW1})^{TT}\right| \sim \left|{}_2\bar{h}^{jk}_{TL}/({}_2\bar{h}^{jk}_{DW1})^{TT}\right| \sim \left(\frac{M}{L}\right)\left(\frac{S^2}{L/\lambda\!\!\!^{-}}\right) << 1 \quad . \tag{66c}$$

VI. SUMMARY OF OUR "PLUG-IN-AND-GRIND" FORMALISM FOR WAVE GENERATION

Our post-linear formalism for wave generation can be summarized as follows:

Regime of Validity. The formalism is valid for any system satisfying these constraints: (i) The gravitational field must be weak everywhere

$$|\bar{h}^{\mu\nu}| << 1 \quad \text{everywhere} \quad , \tag{67a}$$

and the source must be isolated [see discussion preceding eq. (4)]. (ii) Gravitational and nongravitational forces must not balance each other so precisely as to enable second-order gravity to influence the system's motions significantly. (iii) The source must not focus substantially light rays emitted from within itself. Mathematically this constraint says

$$|\alpha(P',P)| << 1 \quad \text{for} \quad P' \text{ any event inside the source,}$$
$$P \text{ any event on the future light cone of } P', \tag{67b}$$

where α is defined by equation (59b). For further discussion of this constraint, see the first half of §IV.C.ii. (iv) The "Shapiro time delay" for light propagation within the source must be small compared to the characteristic timescale $\lambda\!\!\!^{-}$ for internal motions of the source. Mathematically this constraint says that in the mean rest-frame of the source

$$(\Delta t_S)_{\text{internal}} \equiv \gamma(P',P)/|\underset{\sim}{x}-\underset{\sim}{x}'| << \lambda\!\!\!^{-} \quad . \tag{67c}$$

Here $\underset{\sim}{x}$ and $\underset{\sim}{x}'$ are spatial locations of events P and P' that lie inside the source, P is on the future light cone of P', and γ is defined by equation (59d). For further discussion, see the second half of §IV.C.ii.

Calculation of the System's Motion. For a system satisfying these constraints one calculates the internal structure and dynamics by using the post-linear formalism of §IV.B [eqs. (23) and (24)].

Calculation of the Distant Field. To calculate the gravitational field ${}_2\bar{h}^{\mu\nu}$ in the radiation zone, far from the source, one takes the result of the post-linear analysis, plugs it into equations (58) and (59), and grinds.

In paper III we shall use this formalism to calculate gravitational bremsstrahlung radiation.

APPENDIX A

WHY USE THE CURVED-SPACE SCALAR-WAVE OPERATOR?

In laying the foundations of our analysis [in and near eq. (6)] we write the Einstein field equations in terms of the curved-space scalar wave operator $\Box_s$. We choose to do this because the obvious alternatives (the flat-space wave operator $\Box_f$ or the curved-space tensor wave operator $\Box_t$) would ultimately lead to complications or dangers in our analysis.

The flat-space operator $\Box_f$ treats the field propagation from the outset as though it were on flat-space characteristics (straight coordinate lines). Because the true characteristics suffer the Shapiro time delay which involves a logarithm of distance, the use of $\Box_f$ would lead to logarithmic divergences in the radiative field at large r. If one were sufficiently careful, one could remove those divergences without serious error--but that is a dangerous enterprise. Even if one succeeded, one would be left in the end with the interesting effects of focussing, time delay ("transition radiation"), and tail all lumped into the "whump" part of the field. We prefer to keep them separate.

Consider next the curved-space tensor wave operator

$$\Box_t \bar{h}^{\mu\nu} \equiv \bar{h}^{\mu\nu;\alpha}{}_{\alpha} + 2R_{\alpha}{}^{\mu}{}_{\beta}{}^{\nu}\,\bar{h}^{\alpha\beta} - 2R_{\alpha}{}^{(\mu}\bar{h}^{\nu)\alpha} \tag{A1}$$

[cf. MTW eq. (35.64)]. Because the true propagation equation for very weak gravitational waves on a curved background is $\Box_t \bar{h}^{\mu\nu} = 0$, it is tempting to formulate our analysis in terms of $\Box_t$ rather than $\Box_s$. By using $\Box_s$ we push into the "whump" part of ${}_2\bar{h}^{\mu\nu}$ an important physical effect: the curvature-induced rotation of polarization. In effect, part of our whump field corrects the error in our direct field's unrotated polarization. Had we used $\Box_t$ rather than $\Box_s$, polarization rotation would have shown up in §IV.C.iii as a separate piece of the radiation field.

The tensor wave operator has a disadvantage which, for our purposes, outweighs the above advantage. Suppose that one constructed a tensor Green's function for $\Box_t$

$$\Box_t \; G^{\mu\nu\alpha'\beta'}(P',P) = -\tfrac{1}{2}(g^{\mu\alpha'}\; g^{\nu\beta'} + g^{\mu\beta'}\; g^{\nu\alpha'})(gg')^{-1/4}\; \delta_4(x-x') \;, \qquad \text{(A2)}$$

or for <u>any</u> other wave operator with the form

$$\Box_{\text{other}}\; \bar{h}^{\mu\nu} \equiv \bar{h}^{\mu\nu;\alpha}{}_{\alpha} + (\text{any "background" field})^{\mu\nu}{}_{\alpha\beta}\; \bar{h}^{\alpha\beta} \quad . \qquad \text{(A3)}$$

That Green's function would have a first-order tail ${}_1G^{\mu\nu\alpha'\beta'}_{\text{tail}}(P',P)$ with "sources" $\beta^{\mu\nu\alpha'\beta'}$ involving the Riemann tensor [cf. eqs. (43a) and (45c)]. Such a tail would originate everywhere on the light cone of P', whereas the tail ${}_1G^{\text{tail}}$ for our scalar Green's function originates only on rays that have passed through matter. In practical calculations involving lumpy sources--see, e.g., Paper III--that tail would be as difficult to calculate as the whump part of the field. We prefer our scalar tail because of its greater simplicity. By using $\Box_s$ we dump all serious calculational complexities, for lumpy sources, into the whump part of the field.

APPENDIX B

LINE-INTEGRAL IDENTITIES

The weak-field Green's function ${}_1G(P',P)$ used in this paper involves three integrals α,β,γ along "straight lines". In this appendix we take the line of integration to be

$$ {}_0\mathcal{C}(\lambda): \quad \xi^\alpha \equiv x^{\alpha'} + \lambda X^\alpha \ , \qquad 0 \le \lambda \le 1 \ , \quad X^\alpha \equiv x^\alpha - x^{\alpha'} \ . \tag{B1} $$

The three line integrals are

$$ \gamma \equiv \frac{1}{2} X^\mu X^\nu \int_0^1 {}_1h_{\mu\nu} \, d\lambda \quad , \tag{B2} $$

$$ \alpha \equiv \frac{1}{2} X^\mu X^\nu \int_0^1 {}_1R_{\mu\nu} \, \lambda(1-\lambda) \, d\lambda \quad , \tag{B3} $$

$$ \beta \equiv X^\mu X^\nu \int_0^1 {}_1R_{\mu\nu} \, \lambda^2 \, d\lambda \quad , \tag{B4} $$

where ${}_1h_{\mu\nu}$ is assumed to satisfy the deDonder condition

$$ {}_1\bar{h}_{\mu\nu}{}^{,\nu} = {}_1h_{\mu\nu}{}^{,\nu} - \frac{1}{2} \, {}_1h_{,\mu} = 0 \tag{B5} $$

and the Ricci tensor is therefore given by

$$ {}_1R_{\mu\nu} = -\frac{1}{2} \, {}_1h_{\mu\nu,\rho}{}^{\rho} \quad , \tag{B6} $$

and where the index notation used is that of a Lorentz frame in flat spacetime.

Below we list a number of useful identities linking the line integrals α,β,γ, their derivatives at point P, and the values of ${}_1h_{\mu\nu}$ and ${}_1R_{\mu\nu}$ at P:

$$ X^\rho \alpha_{,\rho} = \frac{1}{2} \beta \quad , \tag{B7} $$

$$X^\rho X^\sigma \alpha_{,\rho\sigma} = -\beta + \frac{1}{2} X^\rho X^\sigma \;{}_1R_{\rho\sigma} \quad , \tag{B8}$$

$$X^\rho \gamma_{,\rho} = \gamma + \frac{1}{2} X^\rho X^\sigma \;{}_1h_{\rho\sigma} \quad , \tag{B9}$$

$$X^\rho X^\sigma \gamma_{,\rho\sigma} = X^\rho X^\sigma \;{}_1h_{\rho\sigma} + \frac{1}{2} X^\rho X^\sigma X^\tau \;{}_1h_{\rho\sigma,\tau} \quad , \tag{B10}$$

$$\gamma_{,\rho}{}^{\rho} = -\beta + {}_1h \quad . \tag{B11}$$

Similar identities involving derivatives at P' and mixed derivatives at P' and P can be derived fairly easily. For example,

$$\gamma_{,\rho}{}^{\rho'} = -2\alpha - \frac{1}{2}\,{}_1h - \frac{1}{2}\,{}_1h' \quad , \tag{B12}$$

where ${}_1h \equiv {}_1h(P)$ and ${}_1h' \equiv {}_1h(P')$.

The derivations of these identities are quite straightforward. The necessary techniques are illustrated by the following derivation of identity (B7): By differentiating definition (B3) and making use of equations (B1), we obtain

$$\begin{aligned}
\alpha_{,\rho} &= X^\mu \int_0^1 R_{\mu\rho}\ \lambda(1-\lambda)\ d\lambda \\
&\quad + \frac{1}{2} X^\mu X^\nu \int_0^1 (\partial R_{\mu\nu}/\partial\xi^\sigma)(\partial\xi^\sigma/\partial x^\rho)\ \lambda(1-\lambda)\ d\lambda \\
&= X^\mu \int_0^1 R_{\mu\rho}\ \lambda(1-\lambda)\ d\lambda + \frac{1}{2} X^\mu X^\nu \int_0^1 (R_{\mu\nu,\sigma})(\lambda\delta^\sigma{}_\rho)\ \lambda(1-\lambda)\ d\lambda \\
&= X^\mu \int_0^1 R_{\mu\rho}\lambda(1-\lambda)\ d\lambda + \frac{1}{2} X^\mu X^\nu \int_0^1 R_{\mu\nu,\rho}\ \lambda^2(1-\lambda)\ d\lambda
\end{aligned}$$

Here $R_{\mu\nu,\sigma} = \partial R_{\mu\nu}/\partial\xi^\sigma$ is the derivative of $R_{\mu\nu}$ at the integration point $\mathcal{C}(\lambda)$. When multiplied by X^ρ this expression gives

$$X^\rho \alpha_{,\rho} = X^\mu X^\nu \int_0^1 R_{\mu\nu}\, \lambda(1-\lambda)\, d\lambda + \frac{1}{2} X^\mu X^\nu \int_0^1 (R_{\mu\nu,\rho}\, X^\rho)\, \lambda^2 (1-\lambda)\, d\lambda$$

$$= X^\mu X^\nu \int_0^1 R_{\mu\nu}\, \lambda(1-\lambda)\, d\lambda + \frac{1}{2} X^\mu X^\nu \int_0^1 (dR_{\mu\nu}/d\lambda)\, \lambda^2 (1-\lambda)\, d\lambda \quad .$$

By integrating the last expression by parts we obtain

$$X^\rho \alpha_{,\rho} = X^\mu X^\nu \int_0^1 R_{\mu\nu} [\lambda(1-\lambda) - \tfrac{1}{2}(d/d\lambda)(\lambda^2 - \lambda^3)]\, d\lambda$$

$$= \frac{1}{2} X^\mu X^\nu \int_0^1 R_{\mu\nu}\, \lambda^2\, d\lambda = \frac{1}{2} \beta \qquad \underline{\text{QED}}.$$

In this case the integration by parts gave no endpoint terms; but in other cases [eqs. (B8)-(B12)] nonzero endpoint terms are obtained.

In manipulations of our weak-field Green's function ${}_1G(\mathcal{P}', \mathcal{P})$ and of our second-order gravitational field ${}_2\overline{h}^{\mu\nu}$ (see, e.g., Appendix C) two other identities are useful:

$$\beta\delta' = (\alpha\delta)_{,\rho}{}^{\rho} - \alpha_{,\rho}{}^{\rho}\delta + 4\pi\, \alpha\delta_4(x-x') \quad , \tag{B13}$$

$$(\gamma\delta' + \alpha\delta)_{,\rho}{}^{\rho} = \alpha_{,\rho}{}^{\rho}\delta + {}_1\overline{h}^{\rho\sigma}\, \delta_{,\rho\sigma}$$

$$- [\alpha + \tfrac{1}{2}\, {}_1h + (\gamma_{,\rho} - \tfrac{1}{2} X^\sigma\, {}_1h_{\sigma\rho})\partial^\rho]\, 4\pi\, \delta_4(x-x') \quad . \tag{B14}$$

Here δ is the flat-space propagator between $\mathcal{P}'$ and $\mathcal{P}$

$$\delta \equiv \delta_{ret}(\tfrac{1}{2} X^\rho X^\sigma\, \eta_{\rho\sigma}) \quad , \tag{B15a}$$

which is related to the 4-dimensional Dirac delta function

$$\delta_4(x-x') \equiv \delta(x^0 - x^{0'})\ \delta(x^1 - x^{1'})\ \delta(x^2 - x^{2'})\ \delta(x^3 - x^{3'}) \tag{B15b}$$

by

$$\delta_{,\rho}{}^{\rho} = -4\pi\ \delta_4(x-x') \quad ; \tag{B15c}$$

and δ' is the derivative of the propagator (B15a) with respect to its argument. The absence of primes on indices and on h's in (B13) and (B 14) indicates that all derivatives and endpoint terms are taken at P ; none are at P' . The identities (B13) and (B14) can be derived with some labor from the identities (B7)-(B11).

APPENDIX C

PROOF THAT THE "PLUG-IN-AND-GRIND" FORMULAS FOR ${}_2\bar{h}^{\mu\nu}$ SATISFY THE FIELD EQUATIONS AND GAUGE CONDITION

Here we briefly sketch the proof that our second-order gravitational field [eqs.(58)] satisfies the second-order Einstein field equation [eqs. (16) with $n = 2$] and the deDonder gauge condition ${}_2\bar{h}^{\mu\nu}{}_{,\nu} = 0$. As part of our proof we shall derive expressions for the amount by which each piece of ${}_2\bar{h}^{\mu\nu}$ fails, by itself, to satisfy the field equations and gauge condition.

A preliminary step in our proof is to rewrite the "tail" and "transition" fields (58f) and (58d) in new forms.

Although expression (58f) for the tail seems optimal for practical radiation calculations, the restriction $\mathcal{P}' \in I^-(\mathcal{P})$ makes it nasty for formal manipulations. To get rid of this restriction, we take expression (B13) for $\beta\delta'$, in it we replace $\mathcal{P}$ by $\mathcal{P}''$, and then we insert it into expression (58f). The result,

$$
{}_2\bar{h}^{\mu\nu}_{TL} = (1/\pi) \iint [\alpha(\mathcal{P}',\mathcal{P}'')]_{,\rho''}{}^{\rho''} \, \delta_{ret}(\tfrac{1}{2} X^{\alpha''}X^{\beta''} \eta_{\alpha\beta}) \, \delta_{ret}(\tfrac{1}{2} \overline{X}^{\alpha}\overline{X}^{\beta} \eta_{\alpha\beta})
$$

$$
\times \; {}_2T^{\mu\nu}(\mathcal{P}') \, d^4x'' \, d^4x' \quad , \tag{C1}
$$

is an expression which gives the same value for ${}_2\bar{h}^{\mu\nu}_{TL}$ whether one imposes or omits the restriction $\mathcal{P}' \in I^-(\mathcal{P})$. One way to see that (C1) is oblivious to the restriction $\mathcal{P}' \in I^-(\mathcal{P})$ is this: Take the source equation (31) for the tail of the exact curved-space Green's function; calculate its lowest-order form

$$
G^{tail}{}_{,\rho}{}^{\rho} = -(4\pi)^{-1}[\alpha(\mathcal{P}',\mathcal{P})]_{,\rho}{}^{\rho} \, \delta(\tfrac{1}{2} X^{\alpha}X^{\beta} \eta_{\alpha\beta}) \quad ;
$$

invert this using a flat-space propagator; use the resulting G^{tail} to calculate ${}_2\bar{h}^{\mu\nu}_{TL}$; the result will be expression (C1)--and nowhere in the derivation did one need to impose the restriction $\mathcal{P}' \in I^-(\mathcal{P})$.

Expression (58d) for the transition field involves a "time-delay function" $\gamma(\mathcal{P}',\mathcal{P})$ from which the logarithmic, "external time delay" $\Lambda(-X^\alpha U_\alpha)$ has been removed. A straightforward subtraction of the external time delay is well suited to practical calculations, but poorly suited to formal manipulations of ${}_2\bar{h}^{\mu\nu}$. In the formal manipulations of this appendix we shall perform the truncation in a "smoother" manner: We surround the source by a (hypothetical) cloud of negative-mass material, with total mass, $-M$, equal in magnitude to that of the source, $+M$. We put the cloud far enough from the source (e.g., at radius $\mathcal{L} \sim 100L$) that it is very diffuse, and this contributes negligibly to the line integrals α and β ; but near enough that the Shapiro time delay $2M\,\ln(\mathcal{L}/L)$ in going from source L to cloud $\mathcal{L}$ is small compared to the timescale $\lambda\!\!\!^{-}$ of the source's internal motions. The cloud automatically removes the external Shapiro time delay; no artificial truncation of γ is needed. The second-order gravitational field is then given by equations (58) and (C1) <u>everywhere</u> (inside the source and out), except that we must remove the artificial truncation from (58d):

$$ {}_2\bar{h}^{\mu\nu}_{TR} = 4\int \gamma(\mathcal{P}',\mathcal{P})\ \delta'_{\text{ret}}\left(\tfrac{1}{2}\, X^\alpha X^\beta\ \eta_{\alpha\beta}\right)\ {}_2T^{\mu\nu}(\mathcal{P}')\ d^4x' \quad . \tag{C2} $$

Turn now to the proof that our second-order field satisfies the second-order Einstein field equation. We begin by applying the first-order wave operator

$$ {}_1\Box_s = (\eta^{\alpha\beta} - h^{\alpha\beta})\ \partial_\alpha\partial_\beta \tag{C3} $$

to each of the five pieces of our second-order field. By applying ${}_1\Box_s$ to the direct field (eq. (58b)) and by using equation (B15c) we obtain

$$
{}_1\Box_s \; {}_2\bar{h}_D^{\mu\nu} = -16\pi \left(1 - \frac{1}{2}\, {}_1\bar{h}\right) {}_2T^{\mu\nu}
$$

$$
- 4\bar{h}^{\rho\sigma}\, \partial_\rho \partial_\sigma \int \delta_{ret}\left(\frac{1}{2} X^\alpha X^\beta \eta_{\alpha\beta}\right) {}_2T^{\mu\nu}(P')\, d^4x' \quad . \tag{C4a}
$$

By applying ${}_1\Box_s$ to the whump field (eq. 58e) and by using (B15c) we obtain

$$
{}_1\Box_s \; {}_2\bar{h}_W^{\mu\nu} = -16\pi \; {}_1t^{\mu\nu}_{L-L} - {}_1\bar{h}^{\mu\rho}{}_{,\sigma} \; {}_1\bar{h}^{\nu\sigma}{}_{,\rho} \quad . \tag{C4b}
$$

By applying ${}_1\Box_s$ to the tail field (eq. C1) and by using (B15c) we obtain

$$
{}_1\Box_s \; {}_2\bar{h}_{TL}^{\mu\nu} = -4 \int [\alpha(P',P)]_{,\rho}{}^{\rho}\, \delta_{ret}\left(\frac{1}{2} X^\alpha X^\beta \eta_{\alpha\beta}\right) {}_2T^{\mu\nu}(P')\, d^4x' \, . \tag{C4c}
$$

By applying ${}_1\Box_s$ to the focussing field (58c), and by using (B15c) and the relation $\alpha(P,P) = 0$ [cf. eq. (B3)] we obtain

$$
{}_1\Box_s \; {}_2\bar{h}_F^{\mu\nu} = 4 \int [\alpha(P',P)]_{,\rho}{}^{\rho}\, \delta_{ret}\left(\frac{1}{2} X^\alpha X^\beta \eta_{\alpha\beta}\right) {}_2T^{\mu\nu}(P')\, d^4x'
$$

$$
+ 8 \int [\alpha(P',P)]_{,\rho} \left[\delta_{ret}\left(\frac{1}{2} X^\alpha X^\beta \eta_{\alpha\beta}\right)\right]^{,\rho} {}_2T^{\mu\nu}(P')\, d^4x' \quad . \tag{C4d}
$$

By applying ${}_1\Box_s$ to the transition field (C2), and by using (B14), (B11), (B15c), and limits as $P' \to P$ that are obtainable from (B1)-(B4),we obtain

$$
{}_1\Box_s \; {}_2\bar{h}_{TR}^{\mu\nu} = 4\bar{h}^{\rho\sigma}\, \partial_\rho \partial_\sigma \int \delta_{ret}\left(\frac{1}{2} X^\alpha X^\beta \eta_{\alpha\beta}\right) {}_2T^{\mu\nu}(P')\, d^4x'
$$

$$
- 8 \int [\alpha(P',P)]_{,\rho} \left[\delta_{ret}\left(\frac{1}{2} X^\alpha X^\beta \eta_{\alpha\beta}\right)\right]^{,\rho} {}_2T^{\mu\nu}(P')\, d^4x' \quad . \tag{C4e}
$$

By adding up all five pieces (C4a)-(C4e) we obtain

$$ {}_1\Box_s \; {}_2\bar{h}^{\mu\nu} = -16\pi[(1 - \tfrac{1}{2}\,{}_1\bar{h})\,{}_2T^{\mu\nu} + {}_1t^{\mu\nu}_{L-L}] - {}_1\bar{h}^{\mu\rho}{}_{,\sigma}\,{}_1\bar{h}^{\nu\sigma}{}_{,\rho} \quad , \qquad \text{(C5)} $$

which is the second-order Einstein field equation (16).

Turn now to a proof that our field (58) satisfies the deDonder gauge condition ${}_2\bar{h}^{\mu\nu}{}_{,\nu} = 0$ except for fractional errors of $O(\varepsilon^2)$. From (58b) and the relation

$$ \partial_\nu \int \delta_{ret}(\tfrac{1}{2} X^\alpha X^\beta \eta_{\alpha\beta})\, f(P')\, d^4x' = \int \delta(\tfrac{1}{2} X^\alpha X^\beta \eta_{\alpha\beta})\, \partial_{\nu'} f(P')\, d^4x' \qquad \text{(C6)} $$

valid for any function $f(P')$, we obtain

$$ {}_2\bar{h}_D{}^{\mu\nu}{}_{,\nu} = 4\int \delta_{ret}(\tfrac{1}{2} X^\alpha X^\beta \eta_{\alpha\beta})\, \{{}_2T^{\mu\nu}(P')\,[1 - {}_1\bar{h}(P')]\}_{,\nu'}\, d^4x' \,. \qquad \text{(C7a)} $$

From (58e), (C6), and ${}_1\bar{h}^{\rho\sigma}{}_{,\sigma} = 0$ we obtain

$$ {}_2\bar{h}_W{}^{\mu\nu}{}_{,\nu} = 4\int \delta_{ret}(\tfrac{1}{2} X^\alpha X^\beta \eta_{\alpha\beta})\, [{}_1t^{\mu\nu}{}_{,\nu} + (16\pi)^{-1}\,{}_1\bar{h}^{\mu\rho}{}_{,\sigma\nu}\,{}_1\bar{h}^{\nu\sigma}{}_{,\rho}]_{\text{at } P'}\, d^4x' . $$

(C7b)

We now add (C7a) and (C7b) and use the post-linear equations of motion (24a) rewritten in the form

$$ \left[{}_2T^{\mu\nu}(1 - {}_1\bar{h}) + {}_1t^{\mu\nu}\right]_{,\nu} = 0 $$

[cf. eq. (3)] to obtain

$$ ({}_2\bar{h}_D^{\mu\nu} + {}_2\bar{h}_W^{\mu\nu})_{,\nu} = (4\pi)^{-1} \int \delta_{ret}(\tfrac{1}{2} X^\alpha X^\beta \eta_{\alpha\beta})\, {}_1\bar{h}^{\mu\rho}(P')_{,\sigma'\nu'}\, {}_1\bar{h}^{\nu\sigma}(P')_{,\rho'}\, d^4x' . $$

We then use an integration by parts on $x^{\rho'}$ together with (C6) and a relabelling of indices to obtain

$$({}_2\overline{h}^{\mu\nu}_D + {}_2\overline{h}^{\mu\nu}_W)_{,\nu} = (4\pi)^{-1}\, \partial_\nu \int \delta_{ret}(\tfrac{1}{2}\, X^\alpha X^\beta\, \eta_{\alpha\beta})\ {}_1\overline{h}^{\mu\nu}(P')_{,\sigma'\rho'}\ {}_1\overline{h}^{\rho\sigma}(P')\ d^4x' .$$

We then give P' the new name P'' and rewrite ${}_1\overline{h}^{\mu\nu}(P'')$ as a retarded integral [the solution to eq. (24b)]; the result is

$$({}_2\overline{h}^{\mu\nu}_D + {}_2\overline{h}^{\mu\nu}_W)_{,\nu} = (1/\pi)\ \partial_\nu \iint \delta_{ret}(\tfrac{1}{2}\, \overline{X}^\alpha \overline{X}^\beta\, \eta_{\alpha\beta})\ {}_1\overline{h}^{\rho\sigma}(P'')\ \times$$

$$\times\ [\delta_{ret}(\tfrac{1}{2}\, X^{\alpha''} X^{\beta''}\, \eta_{\alpha\beta})]_{,\rho''\sigma''}\ {}_2T^{\mu\nu}(P')\, d^4x'\, d^4x'' \ . \qquad \text{(C7c)}$$

By applying ∂_ν to expression (C1), adding it onto (C7c), using identities (B14) and (B15c), integrating by parts, and using the limiting forms of (B1)-(B4) as $P \to P'$, we obtain

$$({}_2\overline{h}^{\mu\nu}_D + {}_2\overline{h}^{\mu\nu}_W + {}_2\overline{h}^{\mu\nu}_{TL})_{,\nu} =$$

$$= -4\partial_\nu \int \gamma(P',P)\ \delta'_{ret}(\tfrac{1}{2}\, X^\alpha X^\beta\, \eta_{\alpha\beta})\ {}_2T^{\mu\nu}(P')\ d^4x' \qquad \text{(C7d)}$$

$$-\ 4\partial_\nu \int \alpha(P',P)\ \delta_{ret}(\tfrac{1}{2}\, X^\alpha X^\beta\, \eta_{\alpha\beta})\ {}_2T^{\mu\nu}(P')\ d^4x' \ .$$

Comparison with expressions (C2) for ${}_2\overline{h}^{\mu\nu}_{TR}$ and (58c) for ${}_2\overline{h}^{\mu\nu}_F$ shows that

$$({}_2\overline{h}^{\mu\nu}_D + {}_2\overline{h}^{\mu\nu}_W + {}_2\overline{h}^{\mu\nu}_{TL} + {}_2\overline{h}^{\mu\nu}_{TR} + {}_2\overline{h}^{\mu\nu}_F)_{,\nu} = 0 \ ; \qquad \text{(C8)}$$

i.e., our total second-order field <u>does</u> satisfy the deDonder gauge condition.

REFERENCES

Bird, J. M. 1974, preprint (Cambridge University).

Boughn, S. P., Fairbank, W. M., McAshan, M. S., Paik, H. J., Taber, R. C., Bernat, T. P., Blair, D. G., and Hamilton, W. O. 1974, DeWitt-Morette ed. (1974).

Braginsky, V. B. 1974, in DeWitt-Morette ed. (1974).

Braginsky, V. B. and Nazarenko, V. S. 1971, in Proceedings of the Conference on Experimental Tests of Gravitation Theories, JPL Technical Memorandum 33-499 (Pasadena: Caltech Jet Propulsion Laboratory).

Burke, W. L. 1971, J. Math. Phys., **12**, 402.

Chandrasekhar, S. 1964, Ap. J., **140**, 417.

Chandrasekhar, S. and Esposito, F. P. 1970, Ap. J., **160**, 153.

Chitre, D. M., Price, R. H., and Sandberg V. 1973, **31**, 1018.

______ 1975, Phys. Rev., in press.

Chung, K. P. 1973, Nuovo Cimento, **14B**, 293.

De Witt, B. S. and Brehme, R. W. 1960, Ann. Phys. (USA), **9**, 220.

De Witt-Morette, C. ed. 1974, Gravitational Radiation and Gravitational Collapse, I.A.U. Symposium No.64, D. Riedel Publishers (Dordrecht-Holland).

Einstein, A. 1918, Preuss. Akad. Wiss. Berlin, Sitzber., **1918**, 154.

Ginzburg, V. L. and Frank, I. M. 1946, Zhur. Eksp. Teor. Fiz., **16**, 15.

Havas, P. and Goldberg, J. N. 1962, Phys. Rev. **128**, 398.

John, Von R. W. 1973a, Ann. Phys. (Germany) [7], **29**, 15.

____________ 1973b, ibid, 27.

Kovács, S. J. and Thorne, K. S. 1975, in preparation; cited in text as Paper III.

Landau, L. D. and Lifshitz, E. M. 1951, The Classical Theory of Fields (London: Pergamon Press).

________________________ 1960, ibid, second edition.

Misner, C. W. 1974, in DeWitt-Morette ed. (1974).

Misner, C. W., Thorne, K. S., and Wheeler, J. A. 1973, Gravitation (San Francisco: W. H. Freeman and Co.) , cited in text as MTW.

Peters, P. C. 1966, Phys. Rev., 146, 938.

Press, W. M. 1974, Ann. N.Y. Acad. Sci., 224, 272.

Press, W. M. and Thorne, K. S. 1972, Ann. Rev. Astron. Astrophys., 10, 335.

Rees, M. J. 1974, in Ondes et Radiations Gravitationnelles, Proceedings of International Colloquium No. 220 of the C.N.R.S., Éditions du C.N.R.S., p. 203.

Robaschik, V. D. 1963, Acta Phys. Polon., 24, 299.

Ruffini, R. 1973, in Black Holes, C. De Witt and B. S. De Witt, eds. (New York: Gordon and Breach), p. 451.

Synge, J. L. 1960, Relativity: The General Theory (Amsterdam: North Holland).

Thorne, K. S. 1975, in preparation; cited in text as Paper II .

Thorne, K. S. and Żytkow, A. N. 1975, in preparation.

Weber, J. 1960, Phys. Rev., 117, 307.

________ 1969, Phys. Rev. Letters, 22, 1320.

Zel'dovich, Ya. B. and Polnarev, A. G. 1974, Astr. Zhur., 51, 30; English Transl.: Sov. Astron.-A.J., 18, 17 (1974).

FIGURE CAPTIONS

Fig. 1. The points P, P', P'' used in evaluating the post-linear Green's function ${}_1G(P',P)$ and in calculating the post-linear gravitational-wave field ${}_2\bar{h}^{\mu\nu}(P)$. Part (a) shows the parametrized straight-line curves ${}_0\mathcal{C}(\lambda)$ and ${}_0\mathcal{C}'(\lambda)$ linking P, P', and P''; part (b) shows the 4-vectors X^{μ}, $X^{\mu''}$, and $\bar{X}^{\mu}$ linking them.

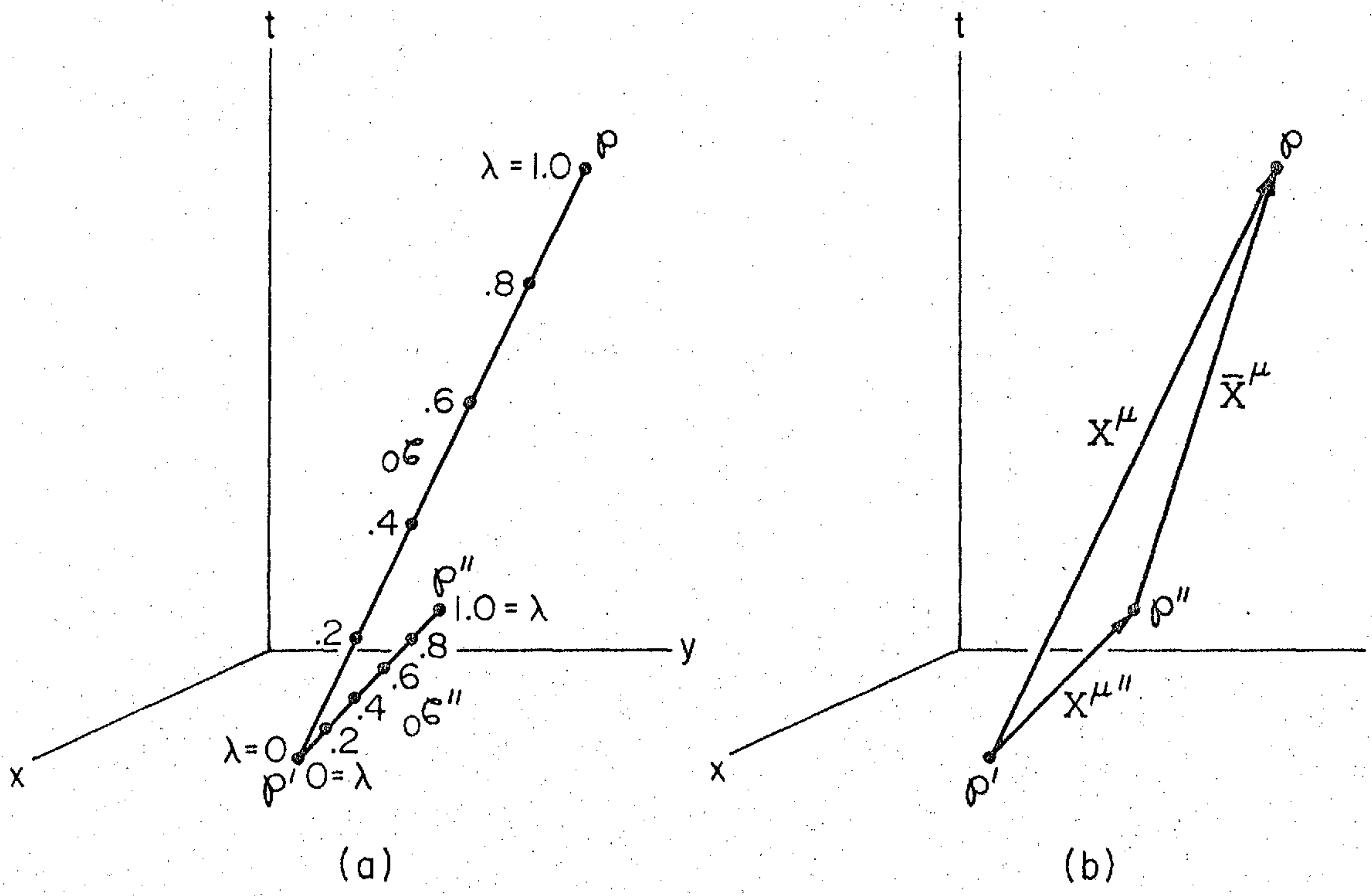

Fig. 1

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